MISSPECIFICATION AND THE EMPIRICAL EVALUATION OF CURVATURE RESTRICTIONS

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Abstract: The rise in popularity of dual models has raised several issues regarding curvature restrictions. This paper proposes a test of curvature restrictions based on the eiginvalues of the Hessian matrix of the dual form. The proposed test is then demonstrated by estimating a dual cost function for agricultural banks.

keywords: curvature, dual functions

1. Introduction

The 1970s spawned a revolution of sorts in empirical analysis of consumer demand relationships and of factor demand and product supply relationships on the part of producers. Two significant developments that may be argued to have played a major role in this revolution were the popularization of what has come to be termed modern microeconomic theory in graduate curricula and the development of flexible functional forms.

Modern microeconomic analysis finds its roots in the writings of Debreu, Hotelling and Shephard, among others, and has been popularized by such books as Cornes, Deaton and Muellbauer, Fuss and McFadden, Varian and Chambers. Some of the most significant features of modern microeconomic theory are its reliance on convex analysis and indirect (dual) objective functions. This, in combination with well known envelope theorems (e.g. Hotelling's and Shephard's lemmas) has led to increased theoretical rigor in the form of mathematical regularity (integrability) conditions on empirical systems of demand (or supply) equations. It should be noted that these developments are especially significant as regards productions theory where for years standard texts treated factor demand and production supply systems virtually as afterthoughts.

Flexible functional forms began their rise to prominence with the seminal articles by Christensen, Jorgenson and Lau who introduced the Translog function and Diewert who developed the Generalized Leontief. Since that time numerous additional flexible functions have been developed including the Fourier (Gallant) and various miniflex forms (Barnett and Lee).

The principle attraction of flexible functional forms rests on their ability to provide a local second order approximation to an arbitrary (true) underlying function. On the surface, this class of functions offers the panacea of avoiding, or at least lessening, concerns over functional specification since they could be viewed as an approximation to the "true" function. However, lost in many discussions and empirical applications is the local nature of the approximation. This is a significant limitation of such functions, an issue recently addressed by Driscoll.

It should be noted that while the most commonly used flexible functions (e.g. Trans-log, generalized Leontief, normalized quadratic) are generally presented as *second order Taylor series approximations*, they may also be considered as *exact* functional specifications. Under such circumstances, these functions are still flexible in the sense that they enable underlying consumer preferences or production structures to be represented without *a priori* restriction. This, as pointed out by Fuss, McFadden and Mundlak, results because the number of unrestricted parameters in commonly used flexible functions coincides with the number of distinct economic effects that characterize unrestricted technology or preference structures.

In combination, the use of indirect objective functions and flexible functional forms have brought systems estimation to the forefront of empirical investigations of demand and supply relationships. Commensurate with this, the imposition of restrictions to ensure "theoretical consistency" have come to be considered standard empirical practice. Many of these theoretical properties such as homogeneity, symmetry, and even homotheticity, may be imposed as uniform linear parameter restrictions in most commonly used specifications. The uniform linear nature of these restrictions provides straightforward global statistical tests of

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conformance of the empirical model and underlying data with these theoretical properties. Indeed, statistical tests of homogeneity and symmetry have received considerable attention (Laitinen; Taylor, et al.; Raj and Taylor).

Some theoretical restrictions, most notably those related to monotonicity and curvature (i.e. concavity and convexity) are not generally testable via global linear parameter restrictions. Because of this, such restrictions are generally "evaluated" in empirical analyses by checking the estimated demand equations for monotonicity and the relevant Hessian matrix for sign (semi-) definiteness at either the mean of the data or at every sample point. While monotonicity is almost always satisfied empirically, the satisfaction of curvature restrictions has been varied; often satisfied at the mean of the data, but seldom satisfied at every sample point.

The absence of a statistical test for curvature restrictions has placed empirical analysts in an unfortunate quandary when simple "evaluations" uncover violations of this property. One can either reject the validity of the model, ignore the violations or rationalize them away, or impose them globally via Cholesky decomposition. Each of these alternatives has deficiencies. Those related to the first alternative are obvious. Simply ignoring curvature violations may lead to inferences that are inconsistent or misleading. Imposing curvature restrictions may seem the best alternatives. However, as shown by Diewert and Wales, globally imposing curvature restrictions on many flexible functional forms has the effect of biasing the second order parameter estimates toward zero, thereby eroding the flexibility of such functions.

Given that curvature restrictions on most indirect objective functions derive from the assumption of optimizing behavior as manifest in the implied convexity of input requirement or indifference sets, *ad hoc* treatment of such properties is not appropriate. Indeed, in the "as if" world that most data aggregations impose on empirical analysis, the ability to statistically evaluate curvature restrictions, and hence the basic assumption of optimizing behavior upon which the very existence of the empirical model rests, is critical. As such, this paper proposes a statistical framework for evaluating curvature restrictions in systems of demand and supply equations.

This paper first examines the definition of concavity linking it with Hermitian matrices. The following section then examines this linkage's implications for duality. Next the paper takes up the topic of estimation and examines how the eigenvalues of a Hermitian matrix can be conditioned on the mean and variance of the estimated parameters. Finally, the paper demonstrates the testing procedure using Featherstone and Moss' data set.

2. Concavity, Hermitian Matrices and the Dual

In order to make our discussion more concrete we focus on the dual cost function. However, the proposed mechanics are fairly general and apply to any indirect objective framework such as the profit function.

The cost function expresses cost as a function of input prices and output levels. Concavity in input prices implies that while cost increases as input prices increase, these increases in cost are smaller for each additional increase in price. From an economic perspective this change in the rate of change implies something about the nature of the production process, the optimizing behavior of producers and substitutability between factors of production.

Mathematically concavity is demonstrated by the Hessian matrix of the cost function. Given any empirical cost specification

$$C = G(x, p) \tag{1}$$

where C is the cost of producing the output vector,x, at a given level of input prices, p. The cost function is concave in prices if and only if

$$dp' \, \nabla^2_{pp} G(x, p) \, dp \; < 0 \tag{2}$$

where dp is any vector of price changes, i.e., that the Hessian matrix be negative definite. The negative definiteness of the matrix can be demonstrated in several ways such as alternating signs of the principle minors, but more direct to the purpose of this study is the use of eigenvalues.

By definition a real valued, symmetric matrix is Hermitian (Lancaster and Tismenetsky, p. 3). Hermitian matrices are somewaht unique among matrices because all the eigenvalues of a real valued Hermitian matrix are also real. Further, a Hermitian matrix is positive (negative) definite if all of its eigenvalues are positive (negative) (Lancaster and Tismenetsky, p 179).

Given that the hessian matrix of the cost function with respect to input prices is symmetric by Young's theorem, the eigenvalues of the Hessian matrix can then be used as a test for concavity. Specifically, if all the eigenvalues of the Hessian matrix with respect to input prices are negative then the cost function specification is concave in input prices. The problem with this definition is that the Hessian matrix will possess as many eigenvalues as it does input prices. Therefore, an alternative measure of concavity is whether the maximum eigenvalue is less than zero. Similarly, testing for convexity in output levels would involve testing whether the minimum eigenvalues of the Hessian matrix with respect to output levels is less than zero.

3. Estimation and the Hermitian

Given the forgoing discussion it is useful to review the typical approach to estimating dual cost functions. First a flexible functional form is specified and parameters estimated using maximum likelihood or three stage least squares. A common approach is to impose homogeneity and symmetry. The estimated values can then be tested for concavity globally in the case of the quadratic function or at some number of points in the case of more general functions. If the Hessian matrix is not negative definite at the estimated values then concavity can be imposed by either estimating the Cholesky decomposition of the Hessian or restricting the eigenvalues of the Hessian matrix to be negative. However, as a first step in this process we propose testing the viability of concavity. If concavity is viable then the researcher can impose concavity reasonably confident in the probability of success and the merits of the restriction. However, if the concavity scenario is not viable then the researcher may want to re-examine his specification.

To compare these scenarios, consider the estimation process. Assuming the parameters are estimated using maximum likelihood, the parameter vector θ is choosen to

$$\begin{aligned} & min_\theta \ \ \tfrac{T}{2} ln|\hat{\Omega}(\theta)| \\ & \hat{\Omega}(\theta) = \textstyle\sum_{t=1}^{T} \hat{\epsilon}_t(\theta)\hat{\epsilon}_t(\theta)' \\ & \hat{\epsilon}_t(\theta) = \begin{bmatrix} C_t - G(x_t, p_t, \theta) \\ z_t^* - \nabla_p^* G(x_t, p_t, \theta) \end{bmatrix} \end{aligned} \tag{3}$$

where $\hat{\Omega}(\theta)$ is the estimated variance matrix, $\hat{\epsilon}_t(\theta)$ is the estimated residual vector for observation t, C_t is the observed cost, $G(x_t, p_t, \theta)$ is the cost function augmented to include the set of estimated parameters, z_t^* denoted sthe n-1 input levels, and $\nabla_p^* G(x_t, p_t, \theta)$ are the n-1 input demand equations. This minimization problem is used to define the parameter estimates and the asymptotic variance of the estimated parameters.

The asymptotic variance of the parameter vector could be used to construct several tests regarding the estimated results. For example, if one were interested in testing an assumption regarding the marginal cost, he could construct a sample of simulated marginal costs by drawing from the sample of θs implied by the estimates. Such tests have been used to compute test statistics for homogeneity and symmetry in small samples.

Extending this concept further, the Hessian matrix of the cost function with respect to input prices can also be derived based on the estimated parameter vector. Thus, the analyst can simulate the Hessian matrix with respect to input prices based on the parameter estimates and the asymptotic variance matrix. Further, the maximum eigenvalue for each draw of the Hessian matrix can be used as a data point to analyze the concavity. Speicifically, for a fixed (x_t, p_t) a sample of Hessian matrices can be constructed and their maximum eigenvalues computed. The sorted sample of maximum eigenvalues then indicates whether concavity can be rejected. For example, if out of 1000 draws 100 eigenvalues are greater than zero the researcher would fail to reject concavity. If, on the other hand 500 maximum eigenvalues were positive he would conclude that the representation was negative semi-definite. However, if 900 eigenvalues out of 1000 were greater than zero the analyst would probably conclude that the sample was *not* consistent with concavity. In the first two scenarios imposing concavity by estimating the Cholesky decomposition or constraining the eigenvalues would appear a reasonable approach while in the third scenario we may interpret te results as indicative of misspecification.

4. An Empirical Example

To demonstrate the use of the criteria we apply the technique to the problem analyzed by Featherstone and Moss. Featherstone and Moss analyze the economies of scope and scale for agricultural banking using a normalized quardatic cost function. The normalized quadratic simplies the discussion regarding the point estimate of concavity since concavity is constant across the specification. Also the data set legitimizes the asymptotic variance matrix because the sample size is fairly large (7,108 banks).

The model estimated is the standard normalized quardratic

$$C = \alpha_0 + \alpha w + \frac{1}{2} w' A w + \beta y + \frac{1}{2} y' B y + y \Gamma w \tag{4}$$

where α_0 is an estimated constant, α and β are estimated vectors, A, B, and Γ are estimated matrices, w is a vector of observed input prices and y is a vector of observed outputs. Equation (4) was estimated using concentrated maximum likelihood imposing symmetry and homogeneity.

One reason this data set was used is that Featherstone and Moss found that concavity in input prices and convexity in output levels do not hold at the estimated values. They then fit the same cost function using the Cholesky decomposition approach suggested by Lau. The question can then be raised regarding the appropriateness of this assumption. Using the maximum likelihood estimates of the A and B matrix and their asymptotic variance matrix, we simulated 1000 draws of the A and B matrix. The average eigenvalue of A was 371.99 with a standard error of 54.85. Similarly, the minimum eigenvalue was -.0000536 with a standard error of .00000476. Thus, if we are willing to accept the normality of the maximum and minimum eigenvalues by the law of large numbers we reject the possibility that the minimum eigenvalues of the A matrix is less than or equal to zero and that the minimum eigenvalue of the B matrix is greater than or equal to zero. This in effect rejects the convexity of the cost function in input prices and the concavity of the cost function in output levels. More telling than the assumption of normality, however, is the fact that none of the maximum eigenvalues of the A matrix are negative and none of the minimum eigenvalues of the B matrix are positive.

These results have numerous consequences for specification. First, any attempt to impose concavity by either methodology will probably prove difficult. Specifically, convergence in the maximum likelihood procedure will be slow and highly dependent on the initial values. More importantly, the original cost function imposing concavity is probably misspecified. For example, the vector of input prices may include quasi-fixed inputs, or, as is more likely the case in the Featherstone and Moss study, the identification of inputs and outputs may be faulty. However, the rejection of concavity may be an indication of a more vexing problem such as the choice of functional forms. For example, Featherstone and Moss choose the normalized quadratic which imposes the same degree of concavity throughout the sample while in fact the concavity may change as banks become larger or smaller. Finally, rejection of concavity may imply poor pooling decisions in constructing the data set.

5. Conclusions

The importance of concavity/convexity conditions in the estimation of dual or indirect objective functions cannot be minimized. If the estimated function violates the optimizing behavior on which it is predicated, little faith can be placed in the economic interpretation of the estimated model. To this end several authors have suggested procedures for imposing concavity on the model. This paper goes one step further by asking whether the failure of concavity is an artifact of misspecification and proposes an empirical test of the reasonableness of concavity. Specifically, the eigenvalues of the Hessian matrix can be sampled to determine the likelihood of concavity. Failure to accept concavity not only provides evidence of misspecification, but also may save the researcher time and resources by ruling out computationally expensive estimation procedures.

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