

Differential Models of Production - Change in Marginal Cost and the Multi-Product Firm: XXIX

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1 Change in Marginal Cost

2 Multiproduct Firm

Change in Marginal Cost

- Shares of Marginal Cost
 - Since both total and marginal cost depend on output levels and input prices, we start by considering marginal share of each input price

$$\theta_i = \frac{\frac{\partial (p_i q_i)}{\partial z}}{\frac{\partial C}{\partial z}} \quad i = 1, \dots, n \quad (1)$$

- Based on this definition, we define a Firsch price index for inputs as

$$d(\ln(P')) = \sum_{i=1}^n \theta_i d(\ln(p_i)) \quad (2)$$

- Completing the single output model

$$d \left[\ln \left(\frac{\partial C}{\partial z} \right) \right] = \left(\frac{\gamma}{\psi} - 1 \right) d \ln(z) + d \ln(P') \quad (3)$$
$$\frac{1}{\psi} = 1 + \frac{1}{\gamma^2} \frac{\partial^2 \ln(C)}{\partial \ln(z) \partial \ln(z)}$$

Multiproduct Firm

- Expanding the production function to a multiproduct technology

$$h(q, z) = 0$$

$$\sum_{r=1}^m \frac{\partial h(q, z)}{\partial \ln(z_r)} = -1 \quad (4)$$

- Expanding the single-product proof

$$L(q, \rho) = \sum_{i=1}^n p_i q_i - \rho h(q, z) \quad (5)$$

- Computing the first-order conditions

$$\frac{\partial L(q, \rho)}{\partial \ln(q_i)} = p_i q_i - \rho \frac{\partial h(q, z)}{\partial \ln(q_i)} = 0 \quad i = 1, \dots, n \quad (6)$$

- Continued

- Now we replicate some of the steps from the preceding lecture, allowing for multiple outputs.
 - Taking the differential of the first-order condition with respect to each output

$$p_i q_i \frac{\partial \ln(q_i)}{\partial \ln(z_r)} - \rho \frac{\partial h(q, z)}{\partial \ln(q_i)} \frac{\partial \ln(\rho)}{\partial \ln(z_r)}$$
$$- \rho \sum_{j=1}^n \frac{\partial^2 h(q, z)}{\partial \ln(q_i) \partial \ln(q_j)} \frac{\partial \ln(q_j)}{\partial \ln(z_r)} - \rho \frac{\partial^2 h(q, z)}{\partial \ln(q_i) \partial \ln(z_r)} = 0$$
(7)

- Again, note by the first-order condition

$$p_i q_i = \rho \frac{\partial h(q, z)}{\partial \ln(q_i)}$$
(8)

- Continued
 - Thus

$$p_i q_i \frac{\partial \ln(q_i)}{\partial \ln(z_r)} - p_i q_i \frac{\partial \ln(\rho)}{\partial \ln(z_r)}$$

$$-\rho \sum_{j=1}^n \frac{\partial^2 h(q, z)}{\partial \ln(q_i) \partial \ln(q_j)} \frac{\partial \ln(q_j)}{\partial \ln(z_r)} - \rho \frac{\partial^2 h(q, z)}{\partial \ln(q_i) \partial (z_r)} = 0 \quad (9)$$

- With

$$f_i = \frac{p_i q_i}{C}$$

$$(F - \gamma H) \frac{\partial \ln(q)}{\partial \ln(z')} - F_\ell \frac{\partial \ln(\rho)}{\partial \ln(z')} = \gamma H^* \quad (10)$$

- Differentiating with respect to input prices yields the same result as before

$$(F - \gamma H) \frac{\partial \ln(q)}{\partial \ln(p')} - F_L \frac{\partial \ln(\rho)}{\partial \ln(p')} = -F \quad (11)$$

- Slightly changing the preceding derivation by differentiating the production function by a vector of output levels, holding prices and other outputs constant yields

$$\sum_{i=1}^n \frac{\partial h(q, z)}{\partial \ln(q_i)} \frac{\partial \ln(q_i)}{\partial \ln(z_r)} + \frac{\partial h(q, z)}{\partial \ln(z_r)} = 0 \quad (12)$$

- Multiplying through by γ yields

$$\sum_{i=1}^n \gamma \frac{\partial h(q, z)}{\partial \ln(q_i)} \frac{\partial \ln(q_i)}{\partial \ln(z_r)} + \gamma \frac{\partial h(q, z)}{\partial \ln(z_r)} = 0 \quad (13)$$

- Continued
 - Using the tired first-order conditions

$$\begin{aligned} p_i q_i - \rho \frac{\partial h(q, z)}{\partial \ln(q_i)} = 0 &\Rightarrow \frac{p_i q_i}{C} - \gamma \frac{\partial h(q, z)}{\partial \ln(q_i)} = 0 \\ &\Rightarrow f_i = \gamma \frac{\partial h(q, z)}{\partial \ln(q_i)} \\ &\Rightarrow \sum_{i=1}^n \gamma \frac{\partial h(q, z)}{\partial \ln(q_i)} \frac{\partial \ln(q_i)}{\partial \ln(z_r)} + \gamma \frac{\partial h(q, z)}{\partial \ln(z_r)} \\ &= \sum_{i=1}^n \gamma f_i \frac{\partial \ln(q_i)}{\partial \ln(z_r)} + \gamma \frac{\partial h(q, z)}{\partial \ln(z_r)} \end{aligned} \tag{14}$$

- Continued
 - With

$$g_r = -\frac{\partial h(q, z)}{\partial \ln(z_r)} = \frac{z_r}{\rho} \frac{\partial C}{\partial z_r}$$

$$\sum_{i=1}^n f_i \frac{\partial \ln(q_i)}{\partial \ln(z_r)} + \gamma \frac{\partial h(q, z)}{\partial \ln(z_r)} = \sum_{i=1}^n \gamma f_i \frac{\partial \ln(q_i)}{\partial \ln(z_r)} - \gamma g_r = 0$$

$$\sum_{i=1}^n \gamma f_i \frac{\partial \ln(q_i)}{\partial \ln(z_r)} - \gamma g_r = 0 \Rightarrow \iota' F \frac{\partial \ln(q)}{\partial \ln(z')} = \gamma g'$$

(15)

- Continued

- Differentiating the production function with respect to yields

$$\sum_{i=1}^n \frac{\partial h(q, z)}{\partial \ln(q_j)} \frac{\partial \ln(q_j)}{\partial \ln(p_i)} = 0 \Rightarrow \iota' F \frac{\partial \ln(q)}{\partial \ln(p')} = 0 \quad (16)$$

- Collecting these equations

- Differentiating the first-order conditions with respect to $\ln(z')$

$$(F - \gamma H) \frac{\partial \ln(q)}{\partial \ln(z')} - F_{\iota} \frac{\partial \ln(\rho)}{\partial \ln(z')} = \gamma H^* \quad (17)$$

- Differentiating the first-order conditions with respect to $\ln(p')$

$$(F - \gamma H) \frac{\partial \ln(q)}{\partial \ln(p')} - F_{\iota} \frac{\partial \ln(\rho)}{\partial \ln(p')} = -F \quad (18)$$

- Continued

- Differentiating the production function with respect to $\ln(z')$

$$\iota' F \frac{\partial \ln(q)}{\partial \ln(z')} = \gamma g' \quad (19)$$

- Differentiating the production function with respect to $\ln(p')$

$$\iota' F \frac{\partial \ln(q)}{\partial \ln(p')} = 0 \quad (20)$$

- System differential

$$\begin{bmatrix} F - \gamma H & \iota' F \\ \iota' F & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \ln(q)}{\partial \ln(z')} & \frac{\partial \ln(q)}{\partial \ln(p')} \\ -\frac{\partial \ln(\rho)}{\partial \ln(z')} & -\frac{\partial \ln(\rho)}{\partial \ln(p')} \end{bmatrix} = \begin{bmatrix} \gamma H^* & -F \\ \gamma g' & 0 \end{bmatrix} \quad (21)$$