

Differential Models of Production - The Single Product Firm: XXVII

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Overview of the Differential Approach

- Until this point we have mostly been concerned with envelopes or variations of deviations from envelopes in the case of stochastic frontier models.
 - The production function was defined as an envelope of the maximum output level that could be obtained from a given quantity of inputs.
 - The cost function was the minimum cost of generating a fixed bundle of outputs based on a vector of input costs.
- The differential approach departs from this basic formulation by examining changes in optimizing behavior.

Consumer Demand

- Starting from consumption theory we have

$$\left. \begin{array}{l} \max U(x) \\ \text{s.t. } p'x \leq Y \end{array} \right\} \Rightarrow \frac{\partial U(x)}{\partial x_i} = \lambda p_i \quad (1)$$

- We assume that consumers choose the levels of consumption so that these first-order conditions are satisfied.
- The question is then what can we learn by observing changes in these first-order conditions or changes in the optimizing behavior.

$$\begin{bmatrix} \frac{\partial U(x)}{\partial x_1} \\ \vdots \\ \frac{\partial U(x)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \lambda p_1 \\ \vdots \\ \lambda p_n \end{bmatrix} \quad (2)$$

Differentiating the First-Order Conditions

$$\begin{aligned}
 & \begin{bmatrix} \frac{\partial^2 U(x)}{\partial x_1 \partial x_2} \frac{\partial x_1}{\partial p_1} + \dots \frac{\partial^2 U(x)}{\partial x_1 \partial x_n} \frac{\partial x_n}{\partial p_1} & \dots & \frac{\partial^2 U(x)}{\partial x_1 \partial x_1} \frac{\partial x_1}{\partial p_n} + \dots \frac{\partial^2 U(x)}{\partial x_1 \partial x_n} \frac{\partial x_n}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 U(x)}{\partial x_n \partial x_1} \frac{\partial x_1}{\partial p_1} + \dots \frac{\partial^2 U(x)}{\partial x_n \partial x_n} \frac{\partial x_n}{\partial p_1} & \dots & \frac{\partial^2 U(x)}{\partial x_n \partial x_1} \frac{\partial x_1}{\partial p_n} + \dots \frac{\partial^2 U(x)}{\partial x_n \partial x_n} \frac{\partial x_n}{\partial p_n} \end{bmatrix} \\
 & \times \begin{bmatrix} dp_1 \\ \vdots \\ dp_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda \end{bmatrix} \begin{bmatrix} dp_1 \\ \vdots \\ dp_n \end{bmatrix} + \begin{bmatrix} p_1 \frac{\partial \lambda}{\partial p_1} & \dots & p_1 \frac{\partial \lambda}{\partial p_n} \\ \vdots & \ddots & \vdots \\ p_n \frac{\partial \lambda}{\partial p_1} & \dots & p_n \frac{\partial \lambda}{\partial p_n} \end{bmatrix} \begin{bmatrix} dp_1 \\ \vdots \\ dp_2 \end{bmatrix} \quad (3)
 \end{aligned}$$

$$U = \begin{bmatrix} \frac{\partial^2 U}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 U}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 U}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 U}{\partial x_n \partial x_n} \end{bmatrix}, \frac{\partial x}{\partial p'} = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \cdots & \frac{\partial x_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial p_1} & \cdots & \frac{\partial x_n}{\partial p_n} \end{bmatrix} \quad (4)$$

$$U \frac{\partial x}{\partial p'} dp = \left(\lambda I + p \frac{\partial \lambda}{\partial p'} \right) dp$$

- Differentiating the income constraint with respect to income

$$\begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial Y} \\ \vdots \\ \frac{\partial x_n}{\partial Y} \end{bmatrix} = 1 \Rightarrow p' \frac{\partial x}{\partial Y} = 1 \quad (5)$$

- Differentiating the income constraint with respect to prices yields

$$\begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \cdots & \frac{\partial x_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial p_1} & \cdots & \frac{\partial x_n}{\partial p_n} \end{bmatrix} \begin{bmatrix} dp_1 \\ \vdots \\ dp_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} dp_1 \\ \vdots \\ dp_2 \end{bmatrix}$$

$$\Rightarrow p' \frac{\partial x}{\partial p'} dp = -x' dp$$

(6)

- To finish the system, we differentiate the first-order conditions with respect to income, yielding

$$\begin{bmatrix} \frac{\partial^2 U}{\partial x_1 \partial x_1} \frac{\partial x_1}{\partial Y} + \cdots \frac{\partial^2 U}{\partial x_1 \partial x_n} \\ \vdots \\ \frac{\partial^2 U}{\partial x_n \partial x_1} \frac{\partial x_1}{\partial Y} + \cdots \frac{\partial^2 U}{\partial x_n \partial x_n} \frac{\partial x_n}{\partial Y} \end{bmatrix} dY = \begin{bmatrix} p_1 \\ \vdots \\ p_2 \end{bmatrix} \frac{\partial \lambda}{\partial Y} dY \quad (7)$$

$$\Rightarrow U \frac{\partial x}{\partial Y} dY = \frac{\partial \lambda}{\partial Y} p dY$$

Barten's Fundamental Matrix

- Putting each of the bits into order, we have Barten's fundamental matrix equation

$$\begin{bmatrix} U & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial p'} \\ -\frac{\partial \lambda}{\partial Y} & -\frac{\partial \lambda}{\partial p'} \end{bmatrix} = \begin{bmatrix} 0 & \lambda I \\ 1 & -x' \end{bmatrix} \quad (8)$$

- Solving Barten's fundamental matrix equation yields

$$\begin{bmatrix} \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial p'} \\ -\frac{\partial \lambda}{\partial Y} & -\frac{\partial \lambda}{\partial p'} \end{bmatrix} = \begin{bmatrix} U & p \\ p' & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \lambda I \\ 1 & -x' \end{bmatrix}$$

$$\frac{\partial x_i}{\partial p_j} = \lambda u^{ij} - \frac{\lambda}{\frac{\partial \lambda}{\partial Y}} \frac{\partial x_i}{\partial Y} \frac{\partial x_j}{\partial Y} - \frac{\partial x_i}{\partial Y} x_j \quad (9)$$

$$w_i d(\ln(q_i)) = \theta_i d(\ln(Q)) + \phi \sum_{j=1}^n \theta_{ij} d\left(\ln\left(\frac{p_j}{p'}\right)\right)$$

Differential Model of Production

- Theil writes the production function in logarithmic space

$$\ln(z) = h(q) \quad (10)$$

- The Cobb-Douglas function then becomes

$$h(q) = a + \sum_{i=1}^n b_i \ln(q_i) \quad (11)$$

- The Lagrange formulation for the logarithmic production function becomes

$$\begin{aligned} L(q, \rho) &= \sum_{i=1}^n p_i q_i + \rho [\ln(z) - h(q)] \\ \Rightarrow \frac{\partial L(q, \rho)}{\partial \ln(q_j)} &= p_j \frac{\partial q_j}{\partial \ln(q_j)} - \rho \frac{\partial h(q)}{\partial \ln(q_j)} = 0 \end{aligned} \quad (12)$$

$$\frac{\partial q_j}{\partial \ln(q_j)} \Rightarrow \frac{1}{\frac{\partial \ln(q_j)}{\partial q_j}} = \frac{1}{\frac{1}{q_j}} = q_j$$

$$\frac{\partial L(p, \rho)}{\partial \ln(q_j)} = p_j q_j - \rho \frac{\partial h(q)}{\partial \ln(q_j)} = 0 \quad (13)$$

- As in the differential demand model, everything has to end up as a share equation, therefore

$$f_j = \frac{p_j q_j}{C} \ni C = \sum_i^n p_i q_i \Rightarrow \frac{\partial L(p, \rho)}{\partial \ln(q_j)} = f_j - \frac{\rho}{C} \frac{\partial h(q)}{\partial \ln(q_j)} = 0$$

$$\rho \equiv \frac{\partial C}{\partial \ln(z)} \Rightarrow \frac{1}{C} \frac{\partial C}{\partial \ln(z)} = \frac{\frac{\partial C}{C}}{\frac{\partial \ln(z)}{\partial \ln(z)}} = \frac{\partial \ln(C)}{\partial \ln(z)} = \gamma \quad (14)$$

$$\begin{aligned}\frac{\partial L(p, \rho)}{\partial \ln(q_j)} &= f_j - \frac{\rho}{C} \frac{\partial h(q)}{\partial \ln(q_j)} = f_j - \gamma \frac{\partial h(q)}{\partial \ln(q_j)} = 0 \\ \Rightarrow \frac{\partial h(q)}{\partial \ln(q_j)} &= \frac{f_j}{\gamma}\end{aligned}\tag{15}$$

- Returning to the earlier expression

$$\begin{aligned}\frac{\partial^2 L(q, \rho)}{\partial \ln(q_j) \partial \ln(q_i)} &\propto \frac{\partial f_j}{\partial \ln(q_i)} - \gamma \frac{\partial^2 h(q)}{\partial \ln(q_j) \partial \ln(q_i)} \\ \Rightarrow F - \gamma H &\ni: H = \left[\frac{\partial^2 h(q)}{\partial \ln(q_j) \partial \ln(q_i)} \right]\end{aligned}\tag{16}$$

$$\frac{\partial \frac{p_j q_j}{C}}{\partial \ln(q_i)} = \frac{p_j}{C} \frac{1}{\frac{\partial \ln(q_i)}{\partial q_j}} = \begin{cases} \frac{p_j q_j}{C} & i = j \\ 0 & i \neq j \end{cases} \quad (17)$$

- A slightly more ominous form of this expression is then

$$\frac{\partial^2 L(q, \rho)}{\partial \ln(q_j) \partial \ln(q_i)} = \delta_{ij} p_i q_i - \rho \frac{\partial^2 h(q)}{\partial \ln(q_j) \partial \ln(q_i)} \quad (18)$$

- Returning to the expression

$$p_i q_i - \rho \frac{\partial h(q)}{\partial \ln(q_i)} \quad (19)$$

- We differentiate with respect to the natural logarithm of the level of output $\ln(z)$, yielding

$$p_i q_i \frac{\partial \ln(q_i)}{\partial \ln(z)} - \rho \frac{\partial h(q)}{\partial \ln(q_i)} \frac{\partial \ln(\rho)}{\partial \ln(z)}$$

$$- \rho \sum_{j=1}^n \frac{\partial^2 h(q)}{\partial \ln(q_i) \partial \ln(q_j)} \frac{\partial \ln(q_j)}{\partial \ln(z)} - \rho \frac{\partial^2 h(q)}{\partial \ln(q_i) \partial \ln(z)} = 0 \quad (20)$$

- Applying the results from the first-order conditions

$$p_i q_i - \rho \frac{\partial h(q)}{\partial \ln(q_i)} = 0 \quad (21)$$

- We get

$$p_i q_i \frac{\partial \ln(q_i)}{\partial \ln(z)} - p_i q_i \frac{\partial \ln(\rho)}{\partial \ln(q_i)} - \rho \sum_{j=1}^n \frac{\partial^2 h(q)}{\partial \ln(q_i) \partial \ln(q_j)} \frac{\partial \ln(q_j)}{\partial \ln(z)} - \rho \frac{\partial^2 h(q)}{\partial \ln(q_i) \partial \ln(z)} \quad (22)$$

- Given $\gamma = \rho/C$

$$(F - \gamma H) \frac{\partial \ln(q)}{\partial \ln(p')} - F_L \frac{\partial \ln(\rho)}{\partial \ln(p')} = -F \quad (23)$$

- Finally, like the demand model, we differentiate the production constraint with respect to output level and input prices.
 - Taking the differential with respect to output level

$$\sum_{i=1}^n \frac{\partial h(q)}{\partial \ln(q_i)} \frac{\partial \ln(q_i)}{\partial \ln(z)} + \frac{\partial h(q)}{\partial \ln(z)} = 0 \quad (24)$$

$$\iota' F \frac{\partial \ln(q)}{\partial \ln(z)} = \gamma$$

- Taking the differential with respect to input the natural logarithm of input prices yields

$$\sum_{i=1}^n \frac{\partial h(q)}{\partial \ln(q_j)} \frac{\partial \ln(q_j)}{\partial \ln(p_i)} = 0 \Rightarrow \iota' F \frac{\partial \ln(q)}{\partial \ln(p')} \quad (25)$$

Barten's Fundamental Matrix for Production

- Putting these bits together

$$\begin{bmatrix} F - \gamma H & \iota \\ \iota' & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \ln(q)}{\partial \ln(z)} & \frac{\partial \ln(q)}{\partial \ln(p')} \\ -\frac{\partial \ln(\rho)}{\partial \ln(z)} & \frac{\partial \ln(\rho)}{\partial \ln(p')} \end{bmatrix} = \begin{bmatrix} \gamma H^* & -F \\ \gamma & 0 \end{bmatrix} \quad (26)$$

- Backing up slightly, we start with

$$(F - \gamma H) \frac{\partial \ln(q)}{\partial \ln(z)} - F \iota \frac{\partial \ln(\rho)}{\partial \ln(z)} = \gamma H^* \quad (27)$$

Pre-multiplying this matrix equation by F^{-1} yields

$$F^{-1} (F - \gamma H) \frac{\partial \ln(q)}{\partial \ln(z)} - \iota \frac{\partial \ln(\rho)}{\partial \ln(z)} = \gamma F^{-1} H^* \quad (28)$$

- Continued

- Next, multiplying the first term by a special form of the identity matrix $F^{-1}F = I$ yielding

$$F^{-1}(F - \gamma H)F^{-1}F \frac{\partial \ln(q)}{\partial \ln(z)} - \iota \frac{\partial \ln(\rho)}{\partial \ln(z)} = \gamma F^{-1}H^* \quad (29)$$

- Using a similar process

$$\begin{aligned} (F - \gamma H) \frac{\partial \ln(q)}{\partial \ln(p')} - F \iota \frac{\partial \ln(\rho)}{\partial \ln(p')} &= -F \\ \Rightarrow F^{-1}(F - \gamma H) \frac{\partial \ln(q)}{\partial \ln(p')} - \iota \frac{\partial \ln(\rho)}{\partial \ln(p')} &= -I \end{aligned} \quad (30)$$

$$\Rightarrow F^{-1}(F - \gamma H)F^{-1}F \frac{\partial \ln(q)}{\partial \ln(p')} - \iota \frac{\partial \ln(\rho)}{\partial \ln(p')} = -I$$

- The matrix - Barten's equation then becomes

$$\begin{bmatrix} F^{-1}(F - \gamma H)F^{-1} & \iota \\ \iota' & 0 \end{bmatrix} \begin{bmatrix} F \frac{\partial \ln(q)}{\partial \ln(z)} & F \frac{\partial \ln(q)}{\partial \ln(p')} \\ -\frac{\partial \ln(\rho)}{\partial \ln(z)} & \frac{\partial \ln(\rho)}{\partial \ln(p')} \end{bmatrix} = \quad (31)$$

$$\begin{bmatrix} \gamma F^{-1} H^* & -I \\ \gamma & 0 \end{bmatrix}$$

$$\begin{bmatrix} F \frac{\partial \ln(q)}{\partial \ln(z)} & F \frac{\partial \ln(q)}{\partial \ln(p')} \\ -\frac{\partial \ln(\rho)}{\partial \ln(z)} & \frac{\partial \ln(\rho)}{\partial \ln(p')} \end{bmatrix} =$$

$$\begin{bmatrix} F^{-1}(F - \gamma H)F^{-1} & \iota \\ \iota' & 0 \end{bmatrix}^{-1} \begin{bmatrix} \gamma F^{-1}H^* & -I \\ \gamma & 0 \end{bmatrix} \quad (32)$$

$$\begin{bmatrix} F^{-1}(F - \gamma H)F^{-1} & \iota \\ \iota' & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \psi(\Theta - \theta\theta') & \theta \\ \theta' & -\frac{1}{\psi} \end{bmatrix}$$

Estimation

$$\Theta = \frac{1}{\psi} F (F - \gamma H)^{-1} F$$

$$\psi = \iota' F (F - \gamma H)^{-1} F \iota$$

$$\theta = \Theta \iota$$

(33)

$$F \frac{\partial \ln(q)}{\partial \ln(p')} = -\psi (\Theta - \theta \theta')$$

$$f_i d \ln(q_i) = \theta_i d \ln(Q) - \psi \sum_{j=1}^n \theta_{ij} d \ln\left(\frac{p_j}{p'}\right)$$