

Measuring Technical Efficiency: Lecture XXVI

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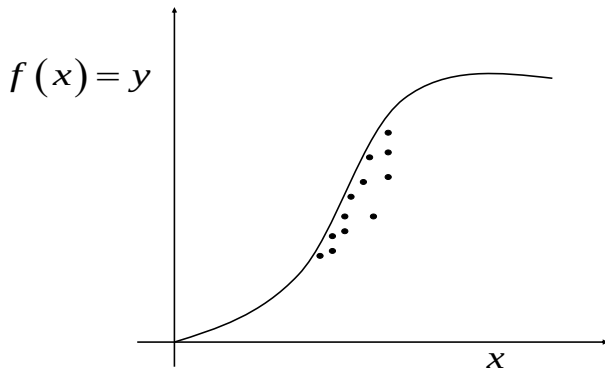
Basic Concepts of Production Efficiency

- Lovell, C. A. Knox. 1993. Production Frontiers and Productive Efficiency. In Harold O. Fried, C. A. Knox Lovell and Shelton S. Schmitz (eds.) *The Measurement of Productive Efficiency*, 3-67.
- The most basic concept of the production function is that they represent some kind of frontier.
 - For example, in our discussion of Diewert, we defined the production function as

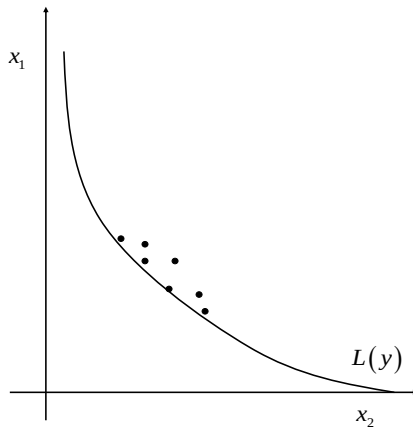
$$f(x) = \max [\mu : x \text{ belongs to } L(\mu)] \quad (1)$$

- Thus, $y = f(x)$ was the largest output possible for a given set of inputs.

Univariate Case



Level Set



- These formulations appear to acknowledge that some firms may be performing sub-optimally. That they could obtain a higher amount of output for the same bundle of inputs.
- This concept underlies the notion of technical inefficiency.
- However, even if the firm is operating on the frontier, we also must recognize that they may be using inputs non-optimally.
 - A basic notion from the production function is that

$$\frac{MPP_1}{MPP_2} = \frac{\frac{\partial f(x)}{\partial x_1}}{\frac{\partial f(x)}{\partial x_2}} = \frac{dx_2}{dx_1} = -\frac{w_1}{w_2} \quad (2)$$

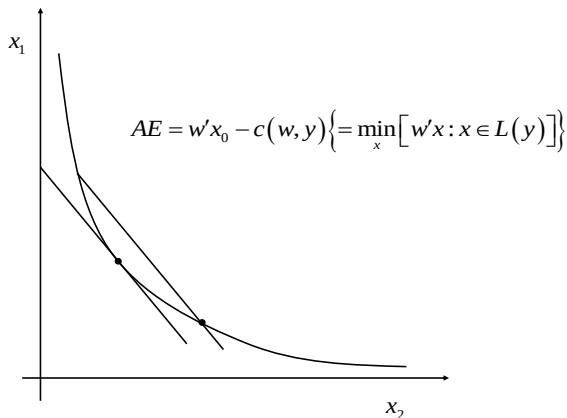
If inputs do not correspond to this allocation, then the firms could trade one input for another and reduce cost.

- Again revisiting our discussion of Diewert

$$c(w, y) = \min_x [w'x : x \text{ belongs to } L(y)] \quad (3)$$

Thus, we have a graphic depiction of the allocative inefficiency

Allocative Inefficiency



$$AE = w'x_0 - c(w, y) \left\{ = \min_x [w'x : x \in L(y)] \right\}$$

The graph shows a coordinate system with axes x_1 and x_2 . A convex set S is represented by a shaded region. A vector x is shown originating from the origin. A supporting hyperplane is drawn tangent to the boundary of S at point x_0 . The vector x is decomposed into x_0 (the efficient component) and $x - x_0$ (the inefficient component). The graph also shows the decomposition of the cost function $c(w, y)$ into its efficient component AE and its inefficient component TE .

A Mathematical Formulation

- Restatement of the level set.
 - Defining the production technology as:

$$L(y) = \{x : (y, x) \text{ is feasible}\} \quad (4)$$

This is the basic production possibility set in Diewert

$$L(y) = [x : f(x) = y, x \text{ nonnegative}] \quad (5)$$

- This also leads to the definition of the isoquant:

$$\text{Iso } L(y) = \{x : x \in L(y), \lambda x \notin L(y), \lambda \in [0, 1]\} \quad (6)$$

- This definition rules out the interior points to the level set.
- The efficient subset can then be defined as

$$\text{Eff } L(y) = \{x : x \in L(y), x' \notin L(y), x' \leq x\} \quad (7)$$

- Based on this definition, the input distance function is written as

$$D_I(y, x) = \max \left\{ \lambda : \left(\frac{x}{\lambda} \right) \in L(y) \right\} \quad (8)$$

- For the isoquant

$$\text{Iso } \mathbb{L}(y) = \{ x : D_I(y, x) = 1 \} \quad (9)$$

- The Debreu-Farrell input oriented measure of technical efficiency can then be expressed as

$$DF_I(y, x) = \min \{ \lambda : \lambda x \in L(y) \}$$

$$DF_I(y, x) \leq 1 \quad (10)$$

$$DF_I(y, x) = \frac{1}{D_I(y, x)}$$

- A slightly different development is given by Fare and Primont
 - In the univariate case

$$F : R_+^N \rightarrow R_+ \\ F(x) = \max_y \{ y : (x, y) \in T \} \quad (11)$$

where T is the technology set. Or

$$T = \{ (x, y) : F(x) \geq y, y \in R_+ \} \quad (12)$$

- The distance function is then given by

$$D_O(x, y) = \min_{\theta} \left\{ \theta : F(x) \geq \frac{y}{\theta} \right\} \quad (13)$$

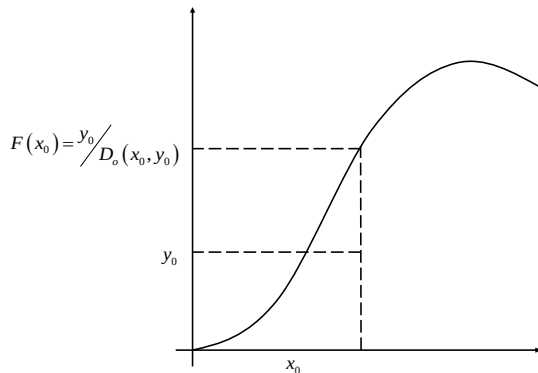
- Alternatively, the distance function can be written in terms of the technology set

$$D_O(x, y) = \min_{\theta} \left\{ \theta : \left(x, \frac{y}{\theta} \right) \in T \right\} \quad (14)$$

- The last representation is then expandable into multivariate space

$$\begin{aligned}
 F : R_+^N &\rightarrow R_+^M & D_O : R_+^N \times R_+^M &\rightarrow R_+ \cup \{+\infty\} \\
 D_O(x, y) &= \inf_{\theta} \left\{ \theta > 0 : \left(x, \frac{y}{\theta} \right) \in T \right\}
 \end{aligned} \tag{15}$$

Fare and Primont



- The Fare-Primont formulation depicts the output augmentation point of view, while the first formulation depicts the input distance formulation.

- $DF_I(y, x)$ is homogeneous of degree -1 in inputs and $DF_O(y, x)$ is homogeneous of degree -1 in outputs.
- $DF_I(y, x)$ is weakly monotonically decreasing in inputs and $DF_O(y, x)$ is weakly monotonically decreasing in outputs.
- $DF_I(y, x)$ and $DF_O(y, x)$ are invariant with respect to changes in units of measurement.
- Measurement with cost and profit functions

$$c(y, w, \beta) = \min_x \{ w'x : D_I(y, x; \beta) \geq 1 \} \quad (16)$$

- General formulation

$$y_i = f(x_i; \beta) \exp \{ \nu_i + \epsilon_i \}$$
$$TE_I = \frac{y_i}{[f(x_i; \beta) \exp \{ \nu_i \}]} = \exp \{ \epsilon_i \} \quad (17)$$

- Econometric Models

- One-sided error terms–Gamma distributions and corrected OLS.
- Composed error term–Stochastic frontier Models.

- Data Envelope Analysis

$$\begin{aligned}
 & \min_z \sum_{i=1}^N z_i c_i = c^* \\
 & \text{s.t.} \quad \sum_{i=1}^N z_i x_{1i} \leq x_1^* \\
 & \quad \quad \quad \vdots \\
 & \quad \quad \quad \sum_{i=1}^N z_i x_{mi} \leq x_m^* \\
 & \quad \quad \quad \sum_{i=1}^N z_i y_{1i} \geq y_1^* \\
 & \quad \quad \quad \vdots \\
 & \quad \quad \quad \sum_{i=1}^N z_i y_{ki} \geq y_k^*
 \end{aligned} \tag{18}$$