

Lecture XXIV: Quasi Fixed Variables and the Revenue Function

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Empirical Revenue Function with Quasi-Fixed Variable

- Suppose we start off with a Mitcherliche-Baul production function

$$y_1 = \beta_{10} (1 - \exp(-\beta_{12}x_1)) (1 - \exp(-\beta_{13}x_2)) (1 - \exp(-\beta_{14}x_3)) \\ \times (1 - \exp(-\beta_{15}x_4))$$

$$y_2 = \beta_{20} (1 - \exp(-\beta_{21}\tilde{y}_1)) (1 - \exp(\beta_{22}x_1)) (1 - \exp(\beta_{23}x_2)) \\ \times (1 - \exp(-\beta_{24}x_3)) (1 - \exp(-\beta_{25}x_4)) \quad (1)$$

where $x_1, \dots, x_4$ are standard inputs and $\tilde{y}_1$ is some quantity of output 1 used to produce output 2 (i.e., corn fed to cattle).

- Given this formulation we maximize profit defined as

$$\begin{aligned} \max_{y,x} p_1 \times (y_1 - \tilde{y}_1) + p_2 y_2 - w'x \\ y_1 - \tilde{y}_1 \geq 0. \end{aligned} \quad (2)$$

- Adding this last constraint to the standard technology notation

$$\left. \begin{aligned} \max_{y,x} p'y - w'x \\ \text{s.t. } \{y, x\} \in T \end{aligned} \right\} \Rightarrow \bar{y}, \bar{x} \quad (3)$$

where $\bar{y}$ denotes the optimal level of observed sales (i.e., we don't observe the amount of corn transferred to livestock feed) and $\bar{x}$ denotes the optimal level of input used.

- Next, I draw a sample assuming the amount of x_4 is quasi-fixed

$$\begin{aligned} \tilde{x}_4 &= \bar{x}_4 \ni: \bar{x}_4 \leq x_4^* \\ \tilde{x}_4 &= \phi (\bar{x}_4 - x_4^*) \ni: \bar{x}_4 > x_4^* \text{ and } 0 \leq \phi < 1 \end{aligned} \quad (4)$$

for some fixed x_4^* .

- Thus Equation 3 is resolved as

$$\left. \begin{aligned} &\max_{y,x} p'y - w'x \\ \text{s.t. } &\{y, x\} \in T, x_4 = \tilde{x}_4 \end{aligned} \right\} \Rightarrow \bar{y}, \bar{x} \quad (5)$$

- Next, we estimate the revenue function

$$R(p, x) = \alpha_0 + \alpha_1 p^* + A_{11} p^{*2} + \beta' x + \frac{1}{2} x' B x + p^* \Gamma x$$

$$y(p, x) = \alpha_1 + A_{11} p^* + \Gamma x$$

$$p^* = \frac{p_1}{p_2} \quad (6)$$

- The results for this estimation are presented in Table 7

Revenue Function Results

Parameter	Estimate	Std. Dev.	t-Ratio	Parameter	Estimate	Std. Dev.	t-Ratio
α_0	872.303	442.867	1.97	B_{14}	-0.209	0.178	-1.17
α_1	3.239	3.601	0.90	B_{22}	-0.213	0.236	-0.90
A_{11}	-3.925	0.166	-23.64	B_{23}	0.122	0.146	0.84
β_1	-0.001	0.096	-0.01	B_{24}	0.527	0.263	2.00
β_2	-31.882	12.020	-2.65	B_{33}	-0.377	0.100	-3.76
β_3	22.139	9.421	2.35	B_{34}	-0.146	0.126	-1.17
β_4	-42.410	19.062	-2.22	B_{44}	1.176	0.464	2.54
B_{11}	-0.220	0.143	-1.54	Γ_{11}	0.181	0.050	3.59
B_{12}	0.578	0.170	3.41	Γ_{12}	0.171	0.053	3.24
B_{13}	-0.045	0.113	-0.40	Γ_{13}	0.240	0.033	7.33
Γ_{14}	0.719	0.072	10.01				

- Given the results in Table 7, we can define the marginal value product of each input

$$\nabla_x R(p, x) \Rightarrow \begin{cases} \beta_1 + B_{11}x_1 + B_{12}x_2 + B_{13}x_3 + B_{14}x_4 + \Gamma_{11}p^* \Rightarrow \hat{w}_1 \\ \beta_2 + B_{12}x_1 + B_{22}x_2 + B_{23}x_3 + B_{24}x_4 + \Gamma_{12}p^* \Rightarrow \hat{w}_2 \\ \beta_3 + B_{13}x_1 + B_{23}x_2 + B_{33}x_3 + B_{34}x_4 + \Gamma_{13}p^* \Rightarrow \hat{w}_3 \\ \beta_4 + B_{14}x_1 + B_{24}x_2 + B_{34}x_3 + B_{44}x_4 + \Gamma_{14}p^* \Rightarrow \hat{w}_4 \end{cases} \quad (7)$$

- Next, we use the results in Equation 7 to test for whether the choice of input levels maximized profit

$$\epsilon = w - \hat{w} \Rightarrow \begin{cases} \frac{\hat{\mu}_{\epsilon i}}{\hat{\sigma}_{\epsilon i}} \\ \hat{\mu}_{\epsilon} \leftarrow E[\epsilon_i] \\ \hat{\sigma}_{\epsilon i} \leftarrow E[(\epsilon - \mu_{\epsilon})'(\epsilon - \mu_{\epsilon})] \end{cases} \quad (8)$$

- These results are presented in Table 9 – profit maximization cannot be rejected except in the case of input 4.

Test for Marginal Value Equal Market Price

Input	Test Statistic
w_1	0.669
w_2	0.762
w_3	0.835
w_4	3.702

Quasi-Fixed Inputs and the Profit Function

- Given our results in Tables 7 and 9 the estimation of the profit function

$$\pi(p, w) = \alpha_0 + \alpha' \begin{pmatrix} p_1^* \\ w_1^* \\ w_2^* \\ w_3^* \\ w_4^* \end{pmatrix} + \frac{1}{2} \begin{pmatrix} p_1^* \\ w_1^* \\ w_2^* \\ w_3^* \\ w_4^* \end{pmatrix}' A \begin{pmatrix} p_1^* \\ w_1^* \\ w_2^* \\ w_3^* \\ w_4^* \end{pmatrix}$$

$$\begin{aligned} y(p, w) &= \alpha_1 + A_{11}p_1^* + A_{12}w_1^* + A_{13}w_2^* + A_{14}w_3^* + A_{15}w_4^* \\ -x(p, w) &= \alpha_2 + A_{12}p_1^* + A_{22}w_1^* + A_{23}w_2^* + A_{24}w_3^* + A_{25}w_4^* \\ -x(p, w) &= \alpha_3 + A_{13}p_1^* + A_{23}w_1^* + A_{33}w_2^* + A_{34}w_3^* + A_{35}w_4^* \\ -x(p, w) &= \alpha_4 + A_{14}p_1^* + A_{24}w_1^* + A_{34}w_2^* + A_{44}w_3^* + A_{45}w_4^* \\ -x(p, w) &= \alpha_5 + A_{15}p_1^* + A_{25}w_1^* + A_{35}w_2^* + A_{45}w_3^* + A_{55}w_4^* \end{aligned} \quad (9)$$

$$p_1^* \frac{p_1}{p_2}, w_1^* = \frac{w_1}{p_2}, \dots, w_4^* = \frac{w_4}{p_2}$$

would appear to be incorrect –

- As an alternative, we formulate the problem as

$$\begin{aligned}\pi(p, w, z) &= p'y = w'x \\ \text{s.t. } \{y, \tilde{x}, z\} &\in T\end{aligned}\tag{11}$$
$$\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad z = x_4$$

- As a starting point, consider the segmentation of inputs from Equation 11 on the revenue function

$$\begin{aligned}
 & \max_y p' y \text{ s.t. } \{y, \tilde{x}, z\} \in T \\
 & R(p, \tilde{x}, z) = \alpha_0 + \alpha_1 p_1^* + \frac{1}{2} A_{11} p_1^{*2} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}' \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} + \\
 & \frac{1}{2} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}' \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} + p_1^* \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} + \\
 & \phi_1 z + z \begin{pmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} + \frac{1}{2} \Phi_{44} z^2 + \gamma_1 p_1^* z \\
 & y(p, \tilde{x}, z) = \alpha_1 + A_{11} p_1^* + \Gamma_{11} \tilde{x}_1 + \Gamma_{12} \tilde{x}_2 + \Gamma_{13} \tilde{x}_3 + \gamma_1 z
 \end{aligned} \tag{12}$$

- Constructing the profit function from the revenue function

$$\pi(p, w) = \max_{\tilde{x}} R(p, \tilde{x}, z) - w' \tilde{x}$$

$$\Rightarrow \beta + B \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \quad (13)$$

$$\Rightarrow \tilde{x}^* = \begin{pmatrix} \tilde{x}_1^* \\ \tilde{x}_2^* \\ \tilde{x}_3^* \end{pmatrix} = B^{-1} \left[\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} - \beta \right].$$

- Substituting the optimal values for inputs into the objective function then yields

$$\begin{aligned}\pi(p, w, z) &= R(p, \tilde{x}^*(p, w, z), z) - w' \tilde{x}^*(p, w, z) \\ y(p, w, z) &= \nabla_p \pi(p, w, z) \\ -x(p, w, z) &= \nabla_w \pi(p, w, z)\end{aligned}\tag{14}$$

- The empirical quasi-fixed profit function then becomes

$$\begin{aligned} \pi(p, w, z) = & \alpha_0 + \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}' \begin{pmatrix} p_1^* \\ w_1^* \\ w_2^* \\ w_3^* \end{pmatrix} + \\ & \frac{1}{2} \begin{pmatrix} p_1^* \\ w_1^* \\ w_2^* \\ w_3^* \end{pmatrix}' \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{12} & A_{22} & A_{23} & A_{24} \\ A_{13} & A_{23} & A_{33} & A_{34} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{pmatrix} \begin{pmatrix} p_1^* \\ w_1^* \\ w_2^* \\ w_3^* \end{pmatrix} + \beta z + \frac{1}{2} B z^2 \quad (15) \\ & + z \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \end{pmatrix} \begin{pmatrix} p_1^* \\ w_1^* \\ w_2^* \\ w_3^* \end{pmatrix} \end{aligned}$$

- Note that the marginal value of the quasi-fixed input can be estimated by taking the derivative of the profit function in Equation 15 with respect to the quasi-fixed variable z .

- The equivalent cost function with quasi-fixed assets can be derived using the formulation

$$\pi(p, w, z) = \max_y [p'y - c(y, w, z)] \quad (16)$$

- Specifically, equating the profit function from Equation 16 with the profit function based on the revenue function

$$K = \max_y [p'y - c(y, w, z)] - \max_x [R(p, x) - w'x] = 0$$

$$\nabla_w K = -\nabla_w c(y, w, z) + x = 0 \Rightarrow x(y, w, z) = \nabla_w c(y, w, z) \quad (17)$$