

Subadditivity of Cost Functions: Lecture XXI

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- Evans, D. S. and J. J. Heckman. 1984. A Test for Subadditivity of the Cost Function with an Application to the Bell System. *American Economic Review* 74, 615-23.
- The issue addressed in this article involves the emergence of natural monopolies.
 - Specifically, is it possible that a single firm is the most cost-efficient way to generate the product.
 - In the specific application, the researchers are interested in the Bell System (the phone company before it was split up).

Mathematics of the Cost Function

- Until this lecture, we have considered cost functions of single outputs.
- The the cost frontier is easily extended to consider multiple outputs

$$\begin{aligned}
 C(w_1, w_2, y_1, y_2) = & \alpha_0 + \alpha_1 w_1 + \alpha_2 w_2 + \\
 & \frac{1}{2} \alpha_{11} w_1 w_1 + \alpha_{12} w_1 w_2 + \alpha_{22} w_2 w_2 + \beta_1 y_1 + \beta_2 y_2 + \\
 & \frac{1}{2} \beta_{11} y_1 y_1 + \beta_{12} y_1 y_2 + \frac{1}{2} \beta_{22} y_2 y_2 \\
 & \gamma_{11} w_1 y_1 + \gamma_{12} w_1 y_2 + \gamma_{21} w_2 y_1 + \gamma_{22} w_2 y_2
 \end{aligned} \tag{1}$$

- The cost function $C(y)$ is subadditive at some output level if and only if

$$\begin{aligned} C(y) &< \sum_{i=1}^n C(y_i) \\ \sum_{i=1}^n q_i &= q \end{aligned} \tag{2}$$

which states that the cost function is subadditive if a single firm could produce the same output for less cost.

- As a mathematical nicety, the point must have at least two nonzero firms. Otherwise the cost function is by definition the same.

- Developing a formal test, Evans and Heckman assume a cost function based on two input

$$\sum_i C(a_i y_1, b_i y_2) > C(y_1, y_2) \quad i = 1, \dots, n$$
$$\sum_i a_i, \sum_i b_i = 1, \quad a_i \geq 0, \quad b_i \geq 0 \quad (3)$$

Thus, each of i firms produce a_i percent of output y_1 and b_i percent of the output y_2 .

- A primary focus of the article is the region over which subadditivity is tested.
 - The cost function is subadditive, and the technology implies a natural monopoly.

$$\sum_i C(a_i y_1, b_i y_2) > C(y_1, y_2). \quad (4)$$

- The cost function is superadditive, and the firm could save money by breaking itself up into two or more divisions.

$$\sum_i C(a_i y_1, b_i y_2) < C(y_1, y_2) \quad (5)$$

- The cost function is additive if

$$\sum_i C(a_i y_1, b_i y_2) = C(y_1, y_2) \quad (6)$$

- The notion of additivity combines two concepts from the cost function: Economies of Scope and Economies of Scale.
- Under Economies of Scope, it is cheaper to produce two goods together.
 - The example I typically give for this is the grazing cattle on winter wheat.
- However, we also recognize following the concepts of Coase, Williamson, and Grossman and Hart that there may diseconomies of scope.
- The second concept is the economies of scale argument that we have discussed before.

Subadditivity and the Translog

- As stated previously, a primary focus of this article is the region of subadditivity.
 - In our discussion of cost functions, I have mentioned the concepts of Global versus local.
 - To make the discussion more concrete, let us return to our discussion of concavity.
 - From the properties of the cost function, we know that the cost function is concave in input price space. Thus, using the Translog form

$$\ln(C) = \alpha_0 + \alpha' \ln(w) + \frac{1}{2} \ln(w)' \ln(w) + \beta' \ln(y) + \frac{1}{2} \ln(y)' B \ln(y) + \ln(w)' \Gamma \ln(y) \quad (7)$$

Gradient

- Continued

- The gradient vector for the Translog cost function is then

$$\begin{aligned}\nabla_w C(w, y) &= \exp[\ln(C)] \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \\ &= \exp[\ln(C)] \begin{bmatrix} \alpha_1 + A_{.q} \ln(w) + \Gamma'_1 y \\ \vdots \\ \alpha_n + A_{.n} \ln(w) + \Gamma'_n y \end{bmatrix}\end{aligned}\quad (8)$$

- Denoting the vector of share equations as $\phi(w, y)$

$$\phi(w, y) = \begin{bmatrix} \alpha_1 + A_{.q} \ln(w) + \Gamma'_1 \ln(y) \\ \vdots \\ \alpha_n + A_{.n} \ln(w) + \Gamma'_n \ln(y) \end{bmatrix}\quad (9)$$

Hessian

$$\begin{aligned}
 \nabla_{xx}^2 C(w, y) &= \exp[\ln(C)] \begin{bmatrix} \alpha_1 + A_{.q} \ln(w) + \Gamma_1 \cdot \ln(y) \\ \vdots \\ \alpha_n + A_{.n} \ln(w) + \Gamma'_n \cdot \ln(y) \end{bmatrix} \\
 &\times \begin{bmatrix} \alpha_1 + A_{.q} \ln(w) + \Gamma_1 \cdot \ln(y) \\ \vdots \\ \alpha_n + A_{.n} \ln(w) + \Gamma'_n \cdot \ln(y) \end{bmatrix}' + \exp[\ln(C)] \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{nn} \end{bmatrix} \\
 &C \times [A + \phi(w, y) \phi(w, y)']
 \end{aligned}
 \tag{10}$$

- Given that the cost is always positive, the positive versus negative nature of the matrix is determined by

$$A + \phi(w, y) \phi(w, y)' \quad (11)$$

- Comparing this results with the result for the quadratic function, we see that

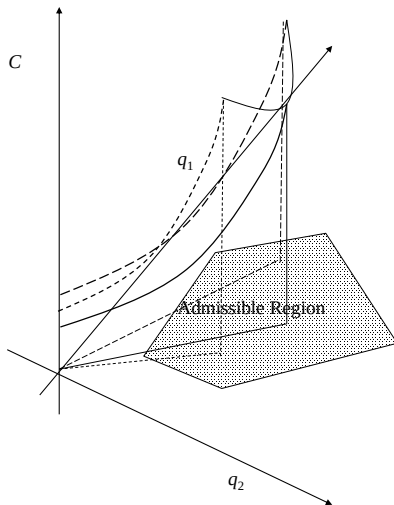
$$\nabla_{ww}^2 C(w, y) = A \quad (12)$$

- Thus, the Hessian of the Translog varies over input prices and output levels while the Hessian matrix for the Quadratic does not.

- In this sense, the restrictions on concavity for the Quadratic cost function are global the do not change with respect to output and input prices.
 - However, the concavity restrictions on the Translog are local fixed at a specific point, because they depend on prices and output levels.
 - Note that this is important for the Translog.
 - Specifically, if we want the cost function to be concave in input prices

$$\begin{aligned}
 x' [A + \phi(w, y) \phi(w, y)'] x &\leq 0 \forall x \\
 \Rightarrow x' Ax + x' \phi(w, y) \phi(w, y)' &\leq 0 \\
 \Rightarrow x' Ax + (\phi(w, y)' x)' (\phi(w, y)' x) &\leq 0
 \end{aligned} \tag{13}$$

- But $(\phi(w, y)' x)' (\phi(w, y)' x) \geq 0$.
- Thus, any discussion of subadditivity, especially if a Translog cost function is used (or any cost function other than a quadratic), needs to consider the region over which the cost function is to be tested.



- Thus, much of the discussion in Evans and Heckman involve the choice of the region for the test.
 - Specifically, the test region is restricted to a region of observed point.
- Defining y_{1m}^* as the minimum amount of y_1 produced by any firm and y_{2m}^* as the minimum amount of y_2 produced, we can define alternative production bundles

$$\begin{aligned}y_t^A &= (\phi y_{1t}^* + y_{1m}, \omega y_{2t}^* + y_{2m}) \\y_t^B &= ((1 - \phi) y_{1t}^* + y_{1m}, (1 - \omega) y_{2t}^* + y_{2m}) \\0 &\leq \phi \leq 1, 0 \leq \omega \leq 1\end{aligned}\tag{14}$$

- Thus, the production for any firm can be divided into two components within the observed range of output.
 - Thus, subadditivity can be defined as

$$\begin{aligned}
 C_t^A(\phi, \omega) &= C(y_t^A) = C(y_m + y_t^A) \\
 C_t^B(\phi, \omega) &= C(y_t^B) = C(y_m + y_t^B) \\
 C_t &= C(y_t^A + y_t^B) = C(y_t) \\
 \text{Sub}_t(\phi, \omega) &= \frac{[C_t - C_t^A(\phi, \omega) - C_t^B(\phi, \omega)]}{C_t}
 \end{aligned} \tag{15}$$

- If $\text{Sub}_t(\phi, \omega)$ is less than zero, the cost function is subadditive, if it is equal to zero the cost function is additive, and if it is greater than zero, the cost function is superadditive
- Consistent with their concept of the region of the test, Evans and Heckman calculate the maximum and minimum $\text{Sub}_t(\phi, \omega)$ for the region.

Composite Cost Functions and Subadditivity

- Pulley, L. B. and Y. M. Braunstein. 1992. A Composite Cost Function for Multiproduct Firms with an Application to Economies of Scope in Banking. *Review of Economics and Statistics* 74, 221-30.
- Building on the concept of subadditivity and the global nature of the flexible function form, it is apparent that the estimation of subadditivity is dependent on functional form
- Pulley and Braunstein allow for a more general form of the cost function by allowing the Box-Cox transformation to be different for the inputs and outputs.

$$\begin{aligned} y^{(\phi)} &= \frac{(y^\phi - 1)}{\phi} \quad \exists: \phi \neq 0 \\ &= \ln(y) \quad \exists: \phi = 0 \end{aligned} \tag{16}$$

- The composite cost function form

$$C^{(\phi)} = \left\{ \exp \left[\alpha_0 + \alpha' q^{(\pi)} + \frac{1}{2} q^{(\pi)'} A q^{(\pi)} + q^{(\pi)} \Psi r \right]^{(\tau)} \right. \\ \left. \exp \left[\beta_0 + \beta' r + \frac{1}{2} r' B r + q^{(\pi)} \Upsilon r \right] \right\}^{(\phi)} \\ f^{(\phi)}(q, \ln(r)) \quad (17)$$

- If $\phi = 0$, $\pi = 0$, and $\tau = 1$ the form yields a standard Translog with normal share equations.
- If $\phi = 0$ and $\tau = 1$ the form yields a generalized Translog

$$\begin{aligned} \ln(C) &= \alpha_0 + \alpha' q^{(\pi)} + \frac{1}{2} q^{(\phi)'} A q^{(\pi)} + q^{(\pi)} \Psi \ln(r) + \\ &\quad \beta' \ln(r) + \frac{1}{2} \ln(r)' B \ln(r) \\ s &= \Psi q^{(\pi)} + \beta + B \ln(r) \end{aligned} \quad (18)$$

- If $\pi = 1$, $\tau = 0$, and $\Psi = 0$, the specification becomes a separable quadratic specification

$$C^{(\phi)} = \left\{ \left[\alpha_0 + \alpha'q + \frac{1}{2}q'Aq \right] \right. \\ \left. \times \exp \left[\beta_0 + \beta' \ln(r) + \frac{1}{2} \ln(r)' B \ln(r) \right] \right\}^{(\phi)} \quad (19) \\ s = \beta + B \ln(r)$$

- The demand equations for the composite function is

$$s = \left[\alpha_0 + \alpha'q + \frac{1}{2}q'Aq + q\Psi \ln(r) \right]^{-1} \Psi q + B \ln(r) + \Upsilon'q \quad (20)$$

Economies of Scope

- Given the estimates, we can then measure Economies of Scope in two ways. The first measures is a traditional measure

$$\text{Scope} = \frac{C(q_1, 0, \dots, 0, r) + \dots C(0, 0, \dots, q_n, r) - C(q_1, q_2, \dots, q_n, r)}{C(q_1, q_2, \dots, q_n, r)} \quad (21)$$

- Another measure suggested by the article is “quasi” economies of scope

$$Q_{\text{scope}} = \frac{C(\{1 - (m-1)\epsilon\} q_1, q_2, \dots, q_n, r) - C(q_1, q_2, \dots, q_n, r)}{C(q_1, q_2, \dots, q_n, r)} \quad (22)$$

- The Economies of Scale are then defined as

$$\text{Scale} = \frac{C(q, r)}{\sum_i q_i \frac{\partial C(q, r)}{\partial q_i}} \quad (23)$$