

Lecture XX: Elasticity of Substitution and Demand

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Table of Contents

1 Elasticities of Substitution

2 Elasticity of Demand

Elasticities of Substitution

- Starting from the Lagrange formulation for cost minimization

$$L = w'x + \lambda (y - f(x)) \Rightarrow \left\{ \begin{array}{l} \frac{\partial L}{\partial x_1} = w_1 - \lambda \frac{\partial f}{\partial x_1} = 0 \\ \vdots \\ \frac{\partial L}{\partial x_n} = w_n - \lambda \frac{\partial f}{\partial x_n} = 0 \\ y - f(x) = 0 \end{array} \right. \quad (1)$$

- Differentiating the first-order conditions then yields

$$\begin{aligned}dw_1 - \lambda \left(\frac{\partial^2 f}{\partial x_1 \partial x_1} dx_1 + \cdots \frac{\partial^2 f}{\partial x_1 \partial x_n} \right) - \frac{\partial f}{\partial x_1} d\lambda &= 0 \\&\dots \\dw_n - \lambda \left(\frac{\partial^2 f}{\partial x_n \partial x_1} dx_1 + \cdots \frac{\partial^2 f}{\partial x_n \partial x_n} \right) - \frac{\partial f}{\partial x_n} d\lambda &= 0 \\dy - \frac{\partial f}{\partial x_1} dx_1 + \cdots \frac{\partial f}{\partial x_n} dx_n &= 0\end{aligned}\tag{2}$$

$$\begin{bmatrix} 0 & -\frac{\partial f}{\partial x_1} & -\frac{\partial f}{\partial x_2} & \cdots & -\frac{\partial f}{\partial x_n} \\ -\frac{\partial f}{\partial x_1} & -\lambda \frac{\partial^2 f}{\partial x_1 \partial x_1} & -\lambda \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & -\lambda \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ -\frac{\partial f}{\partial x_2} & -\lambda \frac{\partial^2 f}{\partial x_1 \partial x_2} & -\lambda \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & -\lambda \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial f}{\partial x_n} & -\lambda \frac{\partial^2 f}{\partial x_n \partial x_1} & -\lambda \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & -\lambda \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix} = \begin{bmatrix} -dy \\ -dw_1 \\ -dw_2 \\ \vdots \\ -dw_n \end{bmatrix}. \quad (3)$$

- Multiplying each row by a form of $(-\lambda) / (-\lambda)$ yields the “Hamermesh” form

$$\begin{bmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \begin{bmatrix} \frac{d\lambda}{\lambda} \\ dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix} = \begin{bmatrix} dy \\ \frac{dw_1}{\lambda} \\ \frac{dw_2}{\lambda} \\ \vdots \\ \frac{dw_n}{\lambda} \end{bmatrix}. \quad (4)$$

- Using Cramer's Rule

$$\begin{bmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \begin{bmatrix} \frac{d\lambda}{d x_1} \\ d x_2 \\ d x_3 \\ \vdots \\ d x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{d w_1}{\lambda} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (5)$$

- Dividing each side by $d w_1$ yields

$$\begin{bmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \begin{bmatrix} \frac{d \lambda}{d w_1} \frac{1}{\lambda} \\ \frac{d x_1}{d w_1} \\ \frac{d x_2}{d w_1} \\ \vdots \\ \frac{d x_n}{d w_1} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\lambda} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (6)$$

- Solving for dx_1/dw_1 by

$$\frac{dx_1}{dw_1} = \frac{\begin{vmatrix} 0 & 0 & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \frac{1}{\lambda} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2} & 0 & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n} & 0 & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{vmatrix}}{\begin{vmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{vmatrix}} \quad (7)$$

$$\frac{dx_1}{dw_1} = \frac{1}{\lambda} \frac{\begin{vmatrix} 0 & 0 & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & 1 & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2} & 0 & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n} & 0 & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{vmatrix}}{\begin{vmatrix} 0 & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{vmatrix}} = \frac{1}{\lambda} \frac{|F_{ij}|}{|F|}. \quad (8)$$

- From our previous development of the elasticity of substitution (in the primal function section):

$$\sigma_{ij} = \frac{\sum_{i=1}^n x_i f_i}{x_i x_j} \frac{|F_{ij}|}{|F|} \text{ where } \frac{|F_{ij}|}{|F|} = \frac{dx_i}{dx_j} \bigg|_{dY=0}$$

- From Equation 8

$$\frac{dx_i}{dw_j} = \frac{1}{\lambda} \frac{|F_{ij}|}{|F|} \Rightarrow \frac{|F_{ij}|}{|F|} = \lambda \frac{dx_i}{dw_j}. \quad (9)$$

- By the first-order conditions $f_i = w_i/\lambda$. Putting these two pieces together

$$\sigma_{ij} = \frac{\sum_{k=1}^n x_k \left(\frac{w_k}{\lambda} \right)}{x_i x_j} \left(\lambda \frac{dx_i}{dw_j} \right) = \frac{\sum_{k=1}^n x_k w_k}{x_i x_j} \frac{\partial^2 c(w, y)}{\partial w_i \partial w_j} \quad (10)$$

- Cleaning up the notation in Equation 10 a little bit

$$\sigma_{ij} = \frac{c(w, y)}{\frac{\partial c(w, y)}{\partial w_i}} \frac{\partial^2 c(w, y)}{\frac{\partial c(w, y)}{\partial w_j} \partial w_i \partial w_j} \quad (11)$$

Elasticity of Demand

- Starting with input 1

$$\zeta_{11} = \frac{\% \Delta x_1}{\% \Delta w_1} = \frac{\partial x_1(w, y)}{\partial w_1} \frac{w_1}{x_1(w, y)}. \quad (12)$$

Substituting the parameters of the general quadratic function into Equation 12 yields

$$\zeta_{11} = A_{11} \frac{w_1}{a_1 + A_{11}w_1 + A_{12}w_2 + A_{13}w_3}. \quad (13)$$

For completeness, consider the response of quantity of input 2 used to a change in the price of input 1

$$\zeta_{21} = A_{12} \frac{w_1}{a_2 + A_{12}w_1 + A_{22}w_2 + A_{23}w_3}. \quad (14)$$