

Limits, Aggregation, and Constraints: Lecture XIX

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1 Basic Functions

- Error in Approximation vs. Residual
- Point of Estimation

2 Aggregation Issues

- Parsimony in Econometric Models

3 Imposing Restrictions

- Homogeneity
- Symmetry
- Concavity

Limitations to Flexible Functional Forms

- The limitations of Flexible Functional Forms, particularly with respect to the limitations imposed by the Taylor series expansion varieties can be demonstrated in several ways.
 - Chambers demonstrates the limitations of the functional forms based on limitations in imposing separability.
 - These arguments are similar to arguments related to imposing separability on various demand systems (i.e. the AIDS models).
- I prefer to demonstrate the limitations to Flexible Function Forms by resorting to the basic notions behind the Taylor Series expansion on which it is based.

Error in Approximation vs. Residual

- Returning to the Taylor series expansion

$$f(x) = f(x_0) + \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0} (x - x_0) + \frac{1}{2} \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_0} (x - x_0)^2 + \cdots \quad (1)$$

$$= \sum_{i=1}^{\infty} \frac{1}{i!} \left. \frac{\partial^i f(x)}{\partial x^i} \right|_{x=x_0} (x - x_0)^i$$

- Truncating the series to the third moment

$$f(x) = f(x_0) + \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0} (x - x_0) + \frac{1}{2} \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_0} (x - x_0)^2 + \frac{1}{6} \left. \frac{\partial^3 f(x)}{\partial x^3} \right|_{x=x_0} (x - x_0)^3 + \text{for some } x^* \in [x, x_0]. \quad (2)$$

- Focusing on the "residual term"

$$\epsilon(x) = \frac{1}{6} \frac{\partial^3 f(x)}{\partial x^3} \bigg|_{x=x_0} (x - x^*)^3 \text{ for some } x^* \in [x, x_0] \quad (3)$$

- As long as the third derivative of the function is non-zero at the point of approximation, we know that the Flexible Functional Form has a "specification" or "approximation" error.
- Further, if we bring this concept together with our typical notions of sampling theory, this approximation error may confound the estimation of parameters.

Point of Estimation

- Finally, there is a problem with the estimation of a functional form and the point of approximation.
 - If one estimates the quadratic cost function, we parameterize the system based on approximations from the arithmetic average.
 - If the Translog is used, the approximation is from the sample's geometric average.
- This raises problems from two perspectives.
 - First, the sample average may not adequately represent a relevant production point.
 - Second, this point of approximation plays into outlier problems.

Aggregation Issues

- Again, issues of aggregation can be addressed at several different levels.
 - One level of aggregation involves the use of a single cost function to depict decisions of numerous farmers.
 - One assumption is that farmers all face similar production functions.
 - Second assuming that all farmers face the same production function, but possess heterogeneous unobserved inputs such as human capital.

- An extension to the heterogeneity issue can be found if we parameterize an aggregate cost function.
 - Capalbo and Denny (AJAE, 1986) examine the impact of changes in technology on U.S. agricultural production using a cost function approach.

$$c(w, y) = \alpha_0 + \alpha'w + \frac{1}{2}w'Aw + \beta'y + \frac{1}{2}y'By + w'\Gamma y + \theta t \quad (4)$$

In this formulation, θ can be used to estimate the impact of changes in technology through time.

- However, to estimate this model we must assume that there exists an aggregate cost function.
 - In other words, we could assume that agriculture in the United States is controlled by a single entity that minimizes cost.
 - Alternatively, we could assume that the minimizing behavior of each individual is the same as an aggregate minimization.

Parsimony in Econometric Models

- As mentioned in earlier lectures, a key element in the estimation of cost functions is parsimony.
 - In general, the number of parameters in a quadratic system is

$$\frac{(n + m + 1)(n + m)}{2} + (n + m) \quad (5)$$

where n is the number of inputs, and m is the number of outputs.

- For accounting purposes in farm level datasets and for degrees of freedom difficulties in when aggregate data is used, we often aggregate inputs and/or outputs.
 - We may aggregate diesel, gasoline, and L.P. gas into a single fuel category.
 - Adding to this we may aggregate fertilizer with fuel to form an agricultural chemical component.
 - In each of these cases, we make fixed factor assumptions between the aggregated inputs.

- Given that these aggregation issues exist, what can be done?
 - One alternative would be to give up on applied work altogether.
 - Another alternative is to use the best data possible, but take a more Bayesian approach. When do the results look right?

- Given the development of the cost function, we are particularly interested in imposing three general conditions on the estimated parameters:
 - Homogeneity,
 - Symmetry, and
 - Concavity.

Homogeneity

- The cost function is homogeneous of degree one in prices and the demand functions are homogeneous of degree zero in prices.
 - The homogeneity restrictions are typically given by

$$\sum_{i=1}^N \alpha_i = 1 \quad \sum_{j=1}^N A_{ij} = 0 \forall i \quad (6)$$

- Back to the original definition, by Euler' theorem, we know that

$$\lambda^t c(w, y) = \sum_{i=1}^N \frac{\partial c(w, y)}{\partial w_i} w_i \quad (7)$$

- Continued

- With the cost function homogeneous of degree one $t = 1$ so

$$1 = \sum_{i=1}^N \frac{\partial c(w, y)}{\partial w_i} \frac{w_i}{c(w, y)} \quad (8)$$

- Or the factor elasticities of the cost function with respect to inputs prices sum to one.
- However, substituting the factor demands from the cost function back into this expression yields

$$1 = \sum_{i=1}^N [\alpha_i + A_{\cdot i} w + \Gamma'_i y] \frac{w_i}{c(w, y)} \quad (9)$$

- Continued

- Let's simplify this structure by choosing both A and Γ to be singular so that $Aw = 0$ for all w and $\Gamma'y = 0$ for all y . In this case Equation 9 becomes

$$1 = \sum_{i=1}^N \frac{\alpha_i w_i}{c(w, y)} \Rightarrow c(w, y) = \sum_{i=1}^N \alpha_i w_i \quad (10)$$

- To guarantee this restriction, we divide through by one of the prices (say w_N) yielding

$$\frac{c(w, y)}{w_N} = \sum_{i=1}^{N-1} \alpha_i \frac{w_i}{w_N} + \alpha_N \rightarrow \alpha_N = \frac{c(w, y)}{w_N} - \sum_{i=1}^{N-1} \alpha_i \frac{w_i}{w_N} \quad (11)$$

- For the Translog approximation, the well known

$$\alpha_N = (\ln(c(w, y)) - \ln(w_N)) - \sum_{i=1}^{N-1} \alpha_i (\ln(w_i) - \ln(w_N))$$

Symmetry

- Symmetry is a standard linear restriction.

$$A_{ij} = A_{ji} \forall i, j \quad (13)$$

Concavity

- As we have discussed concavity is a result of optimizing behavior.
- If the cost-function is not concave, then taking linear combination in price space could further reduce cost.
- Thus, a non-concave cost function is inconsistent with economic theory.
- Two problems: Imposing concavity and rejecting concavity.

Three Approaches to Imposing Concavity

- Two methods impose concavity with an equality
 - The Lau Decomposition (Featherstone and Moss [1994])

$$x'Ax = x'P'Px = (Px)'(Px) \geq 0 \quad (14)$$

This makes the estimation very nonlinear

$$P = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \quad (15)$$
$$\Rightarrow P'P = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{12} & a_{22} & 0 \\ a_{13} & a_{23} & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

- Continued
 - Continuing the matrix operation

$$P'P = \begin{bmatrix} a_{11}a_{11} & a_{11}a_{12} & a_{11}a_{13} \\ a_{11}a_{12} & a_{12}a_{12} + a_{22}a_{22} & a_{12}a_{13} + a_{22}a_{23} \\ a_{11}a_{13} & a_{12}a_{13} + a_{22}a_{23} & a_{13}a_{13} + a_{23}a_{23} + a_{33}a_{33} \end{bmatrix} \quad (16)$$

- The alternative approach is to use the fact that a positive definite symmetric matrix has all positive eigenvalues and a negative definite symmetric matrix has all negative eigenvalues.
 - Thus, we could simply constrain

$$\max [\lambda_i] \leq 0 \ni: \lambda = \text{eigenvalue (A)} \quad (17)$$

- The non-singular method is by bootstrapping the estimation and retaining the estimates that obey Equation 17.