

Cost Functions and the Estimation of Flexible Functional Forms: Lecture XVIII

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Flexible Functional Forms

- The crux of the dual approach is then to estimate a manifestation of behavior that economist know something about.
 - Thus, instead of estimating production functions that are purely physical forms that economist have little expertise in developing, we could estimate the cost function that represents cost minimizing behavior.
 - We then would be able to determine whether the properties of these cost functions are consistent with our hypotheses about technology.

- However, it is often the direct implications of the cost minimizing behavior that we are interested in:
 - How will farmers react to changes in agricultural prices through commodity programs?

$$\frac{\partial C(y; w)}{\partial y} = p + \Delta p \quad (1)$$

- What is the impact of a change in input prices (say in an increase in fuel prices) on agricultural output?

$$\frac{\partial C(y; w + \Delta w)}{\partial y} = p \quad (2)$$

- Thus, the dual cost function results are usually sufficient for most question facing agricultural economists.

- Given that we are interested in estimating the cost function directly, the next question involves how to specify the cost function?
 - One approach to the estimation of cost functions would then be to hypothesize a primal production function and derive the theoretically consistent specification for the cost function based on this primal.
 - However, this approach would appear too restrictive.
 - Thus, economists have typically turned to flexible functional forms that allow for a wide variety of technologies.

Approximation Theory

- A basic approach to the specification of a cost function is to assume that an unspecified function exists, and then derive a closed form approximation of the function.
 - One typical approach from optimization theory involves the Taylor series expansion:

$$\begin{aligned} f(x) &= f(x_0) + \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0} (x - x_0) + \frac{1}{2} \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_0} (x - x_0)^2 + \dots \\ &= \sum_{i=1}^{\infty} \frac{1}{i!} \left. \frac{\partial^i f(x)}{\partial x^i} \right|_{x=x_0} (x - x_0)^i \end{aligned} \quad (3)$$

- Continued
 - To make the expansion finite

$$f(x) = f(x_0) + \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0} (x - x_0) + \frac{1}{2} \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_0} (x - x_0)^2 + \frac{1}{6} \left. \frac{\partial^3 f(x)}{\partial x^3} \right|_{x=x_0} (x - x^*)^3$$

for some $x^* \in [x, x_0]$

(4)

- The real problem is that we don't know the value of x^* .
- As a result, we approximate this term with a residual

$$f(x) = f(x_0) + \left. \frac{\partial f(x)}{\partial x} \right|_{x=x_0} (x - x_0) + \frac{1}{2} \left. \frac{\partial^2 f(x)}{\partial x^2} \right|_{x=x_0} (x - x_0)^2 + \epsilon (x - x_0)^3$$
(5)

Quadratic

- Continued
 - Given this approach, we can conjecture the relative size of the approximation error based on the relative size of the third derivative of the cost function.
 - Extending this result to vector space, we have

$$f(x) = f(x_0) + \nabla_x f(x_0)(x - x_0) + \frac{1}{2}(x - x_0)' \nabla_{xx} f(x_0)(x - x_0) + \epsilon(x - x_0) \quad (6)$$

- Given that the cost function is a function of input prices w and output levels y , we could then stack the two into a single vector and derive the flexible functional form

$$C(w, y) = \alpha_0 + \alpha'w + \frac{1}{2}w'Aw + \beta'y + \frac{1}{2}y'By + w'\Gamma y + \epsilon \quad (7)$$

- Continued
 - This form is typically referred to as the quadratic cost function. It is a second-order Taylor series expansion to an unknown cost function.
 - Following the general concept (and ignoring for the moment the error of approximation), Shephard's lemma can be applied to this cost specification

$$\frac{\partial C(w, y)}{\partial w_i} = x_i^*(w, y) = \alpha_i + A_i \cdot w + \Gamma'_i y. \quad (8)$$

Flexible Cost System

- 4 input - 2 output cost system

$$C(w, y) = \alpha_0 + \alpha'w + \frac{1}{2}w'Aw + \beta'y + \frac{1}{2}y'By + w'\Gamma y$$
$$x_1 = \alpha_1 + A_{11}w_1 + A_{12}w_2 + A_{13}w_3 + A_{14}w_4 + \Gamma_{11}y_1 + \Gamma_{21}y_2$$
$$x_2 = \alpha_2 + A_{12}w_1 + A_{22}w_2 + A_{23}w_3 + A_{24}w_4 + \Gamma_{12}y_1 + \Gamma_{22}y_2$$
$$x_3 = \alpha_3 + A_{13}w_1 + A_{23}w_2 + A_{33}w_3 + A_{34}w_4 + \Gamma_{13}y_1 + \Gamma_{23}y_2$$
$$(9)$$

- Why have I imposed symmetry?
- Why am I only estimating three demand curves?

- One generalization of the Taylor series approach involves a transformation of variables.
- Specifically, if we assume

$$g(x) = \frac{x^\lambda}{\lambda} \quad (10)$$

- The cost function can then be expressed as

$$\begin{aligned} g[C(w, y)] = & \alpha_0 + \alpha' g[w] + \frac{1}{2} g[w]' A g[w] + \beta' g[y] \\ & + \frac{1}{2} g[y]' B g[y] + g[w]' \Gamma g[y] \end{aligned} \quad (11)$$

- This formulation complicates the Shephard's lemma results slightly

$$\frac{\partial \frac{[C(w, y)]^\lambda}{\lambda}}{\partial w} = [C(w, y)]^{\lambda-1} \frac{\partial C(w, y)}{\partial w}$$
$$\frac{\partial \left[\frac{w^\lambda}{\lambda} \right]}{\partial w} = w^{\lambda-1} \quad (12)$$

$$\Rightarrow \frac{\partial \frac{[C(w, y)]^\lambda}{\lambda}}{\partial \left[\frac{w^\lambda}{\lambda} \right]} = \left[\frac{C(w, y)}{w} \right]^{\lambda-1} \frac{\partial C(w, y)}{\partial w} = \left[\frac{C(w, y)}{w} \right]^{\lambda-1} x^*$$

Translog

- Note that letting λ go to zero implies

$$\lim_{\lambda \rightarrow 0} \frac{x^\lambda}{\lambda} = \ln [x] \quad (13)$$

$$\lim_{\lambda \rightarrow 0} \left[\frac{C(w, y)}{w} \right]^{\lambda-1} x^* = \frac{w_i x_i}{C(w, y)} = s_i$$

- In genral
 - $\lambda = 0 \Rightarrow$ Translog.
 - $\lambda = 1 \Rightarrow$ Quadratic.
 - $\lambda = 1/2 \Rightarrow$ Leontief.

Fourier Expansion

- In a univariate sense, any function can be approximated by a series of sine and cosine curves

$$f(x) = \alpha_0 + \sum_{i=1}^N \left[\alpha_i \sin\left(\frac{2\pi x}{\lambda_i}\right) + \beta_i \cos\left(\frac{2\pi x}{\lambda_i}\right) \right] \quad (14)$$

where the λ_i are different periodicities.

- Extending this representation to a multivariate formulation

$$f(x) = \alpha_0 + \sum_{i,\alpha} [\alpha_i \sin(\lambda_i k'_\alpha x) + \beta_i \cos(\lambda_i k'_\alpha x)] \quad (15)$$

where λ_i is a constant for periodicity i and k_α is referred to as an Elementry Multi-Index

$$\left\{ k : |k|^* = \sum_{i=1}^N |k_i| \leq K \right\} \quad (16)$$

- Taking for example the above 4 input - 2 output specification

$$z = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ y_1 \\ y_2 \end{bmatrix} \quad (17)$$

- $K = 1$

$$k_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, k_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \dots k_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (18)$$

- $K = 2$

$$\begin{aligned} k_7 &= \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, k_8 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, k_9 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \dots \\ k_{12} &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, k_{13} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, k_{14} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \dots \end{aligned} \tag{19}$$

- The full form of the multivariate Fourier is then

$$C(z) = \alpha_0 + \alpha'z + \frac{1}{2}z'Az + \sum_{i,\tilde{\alpha}} \left[\tilde{\alpha}_{i,\alpha} \sin(\lambda_i k'_{\tilde{\alpha}} z) + \tilde{\beta}_{i,\alpha} \cos(\lambda_i k'_{\tilde{\alpha}} z) \right] \quad (20)$$

$$A = \sum_{i,\tilde{\alpha}} k_{\tilde{\alpha}} k'_{\tilde{\alpha}}$$

- This representation minimizes the Sobolev Norm, which says that it does a better job approximating the derivatives of the function.
 - In fact, it represents up to the k^{th} derivative of the function.
- Note that if the cost function is specified as a multivariate Fourier expansion, the system of demand equations can be defined by Shephard's lemma.

Estimation of Cost Systems

- Regardless of the function form, cost functions are typically estimated as systems of equations using
 - Seemingly Unrelated Regression,
 - Iterated Seemingly Unrelated Regression, or
 - Maximum Likelihood.
- I prefer the use of Maximum Likelihood based on a concentrated likelihood function.
 - Defining the error term as

$$\epsilon_i(\theta) = \begin{bmatrix} C_i - C(w_i, y_i) \\ x_{1,i} - x_1(w_i, y_i) \\ \vdots \\ x_{n-1,i} - x_{n-1}(w_i, y_i) \end{bmatrix} \quad (21)$$

- Continued
 - Log-Likelihood function

$$L \propto -\frac{N}{2} \ln |\Omega| - \frac{1}{2} \sum_{i=1}^N \epsilon_i(\theta)' \Omega^{-1} \epsilon_i(\theta)$$

$$\Rightarrow L \propto -\frac{N}{2} \ln |\Omega^*(\theta)| \ni \Omega^*(\theta) = \frac{1}{N} \sum_{i=1}^N \epsilon_i(\theta) \epsilon_i(\theta)'$$

$$\nabla_{\theta} L = - \sum_{i=1}^N \epsilon_i(\theta)' \Omega^{-1}(\theta) \nabla_{\theta} \epsilon_i(\theta)$$

(22)