

Shepard's Duality Proof - Part II: Lecture XVII

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1 Basic Functions

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Basic Functions

- Following Shephard's development from the last lecture, we have two basic groups of functions:
 - The distance function, production function, and associated level set.
 - The level set $L_{\Phi}(u)$ is defined as the set of possible combinations of inputs that can be used to produce the output level u .
 - Given the level set, we can define a distance function as

$$\Psi(u, x) = \frac{||x||}{||\lambda_0 x||} \ni \lambda_0 = \min \{ \lambda \mid \lambda x \in L_{\Phi}(u) \} \quad (1)$$

- The production function can then be defined as

$$\Phi(x) = \max \{ u \mid \Psi(u, x) \geq 1 \} \quad x \in D \quad (2)$$

- In addition, we can define the set of efficient input vectors by the distance function

$$E(u) = \{ x \mid \Psi(u, x) = 1 \} \quad (3)$$

- The function and the cost structure can be defined based on the cost-minimization problem.
 - The cost function is defined as

$$Q(u, p) = \min_x \{ p'x \mid x \in L_\Phi(u) \} \quad (4)$$

- The cost structure is then defined for the set of all possible input prices in a similar way as the level sets of inputs are defined in output space. Specifically,

$$\Lambda_Q(u) = \{ p \mid Q(u, p) \geq 1, p \geq 0 \} \quad (5)$$

- With the equality $Q(u, p) = 1$ being established by the normalization of

$$\frac{p}{\|p_0\|} \quad (6)$$

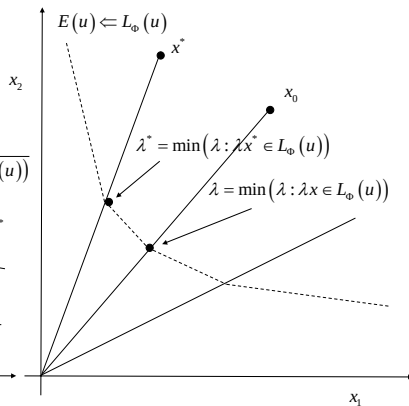
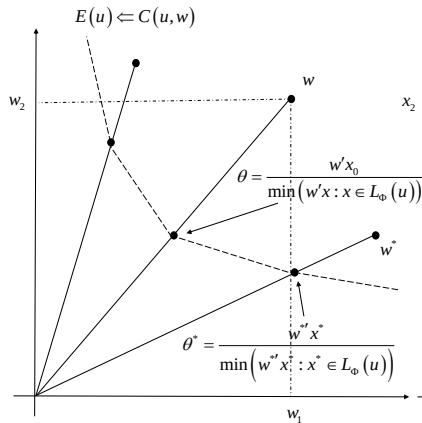
- By symmetry, we can define a set of efficient prices for $E(u)$.

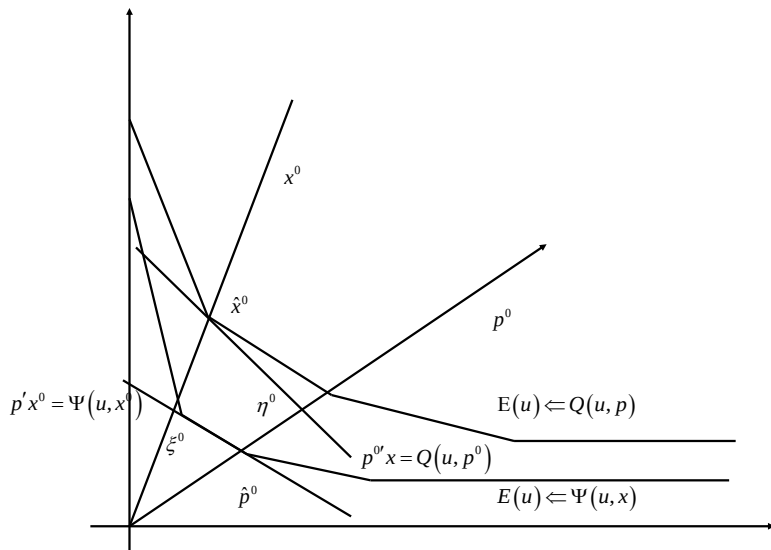
Shepard Duality

- The claim of Shepard duality is then

$$\begin{aligned} Q(u, p) &= \min_x \{ p'x \mid \Psi(u, x) \geq 1 \} , u \geq 0, p \geq 0 \\ \Psi(u, x) &= \inf_p \{ p'x \mid Q(u, p) \geq 1 \} , u \geq 0, x \geq 0 \end{aligned} \quad (7)$$

- Geometric Relationship between Dual Cost and Production Structures
 - “There is a rather simple and elegant geometric relationship between the isoquants of the sets $L_\Phi(u)$ of the production structure and the isoquants of the sets $\Lambda_Q(u)$ of the cost structure, for any positive output rate u .”
 - The proof of the correspondence involves showing the relationship between the efficient sets $E(u)$ defined from the distance function and $E(u)$ defined from the cost function.





- That is that a price vector p^0 and a level of output u determine the level of input use $\hat{x}_0$ on the cost surface $Q(u, p^0)$.
- Put differently

$$\hat{x}_0 = Q(u, p^0) \quad (8)$$

- This point is not necessarily unique.

- Based on this relationship, we define the ray

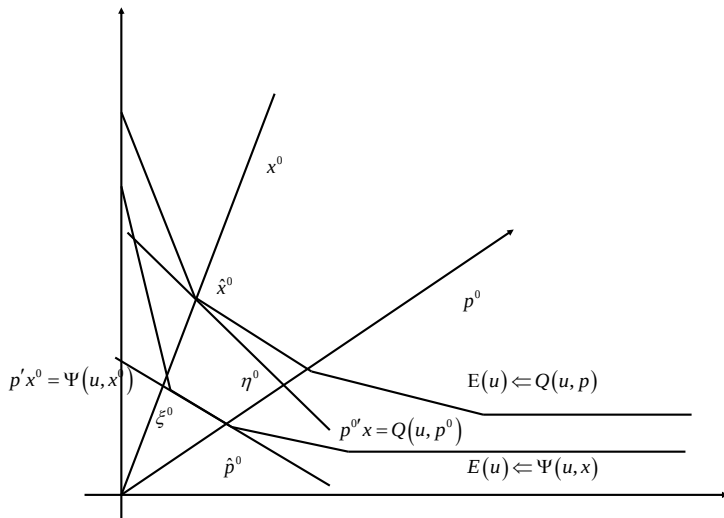
$$\hat{p}_0 = \frac{p^0}{Q(u, p^0)} \quad (9)$$

- Working backwards

$$Q(u, \hat{p}_0) = Q\left(u, \frac{p^0}{Q(u, p^0)}\right) = 1 \quad (10)$$

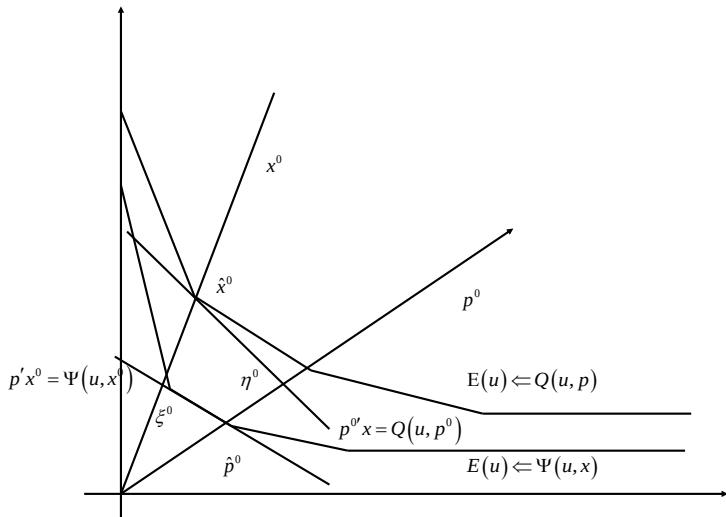
- $\hat{p}_0 \in \Lambda_Q(u)$ since the cost function is the distance function for set $\{Q(u)\}$. Further,

$$\|\hat{p}^0\| = \frac{\|p^0\|}{Q(u, p^0)} \quad (11)$$



- Letting η_0 be the intersection between the cost function at (u, p^0) and the ray $\{ \theta p^0 \mid \theta \geq 0 \}$

$$\begin{aligned}
 ||\hat{p}^0|| \mid ||\theta p^0|| &= Q(u, p^0) \Rightarrow \theta = \frac{Q(u, p^0)}{||p^0||^2} \\
 \Rightarrow ||\eta_0|| &= \frac{Q(u, p^0)}{||p^0||} = \frac{1}{||\hat{p}_0||} \quad (12) \\
 p^{0'} x &= Q(u, p^0)
 \end{aligned}$$

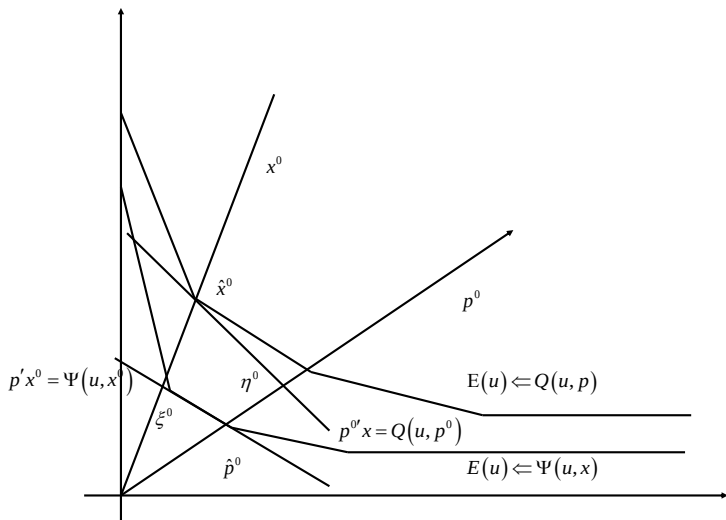


- Working from the other side, we define support of $\Lambda_Q(u)$ as

$$x^0 p = \Psi(u, x^0) \quad (13)$$

In this direction, $\hat{p}_0$ is the contact point between the ray defined by x^0 and the level set $\Psi(u, x^0)$.

$$\hat{x}^0 = \frac{x^0}{\Psi(u, x^0)} \quad (14)$$



- Therefore

$$x^{0'} \hat{p}_0 = \Psi(u, x^0) \Rightarrow \hat{x}_0 p^0 = 1 \quad (15)$$

- Because

$$p^{0'} \hat{x}_0 = Q(u, p^0) \Rightarrow \Psi(u, \hat{x}^0) = 1$$

$$||\hat{x}^0|| = \frac{||x^0||}{\Psi(u, x^0)} \quad (16)$$

- Back to the distance function representations, we let ξ_0 be the intersection of the ray $\{ \lambda x^0 \mid \lambda \geq 0 \}$ with the hyperplane $x^{0'} p = \Psi(u, x^0)$.
- Hence,

$$||x^0|| ||\lambda^0 x^0|| = \Psi(u, x^0) \Rightarrow \lambda^0 = \frac{\Psi(u, x^0)}{||x^0||^2}$$

$$\Rightarrow ||\xi^0|| = \lambda^0 ||x^0|| = \frac{\Psi(u, x^0)}{||x^0||} = \frac{1}{||\hat{x}^0||} \quad (17)$$