

Shephard's Duality Proof - Part I: Lecture XVI

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1 Definiton of the Distance Function

Definition of the Distance Function

- As a starting point of the discussion of the distance function, Shephard defines the space of possible inputs over the nonnegative domain D .
 - Then the space of all possible input bundles is segmented into a sequence of regions:
 - The origin: $\{0\}$
 - Interior points of D : $D_1 = \{x | x > 0\}$
 - The boundary points of D excluding the origin:

$$D_2 = \left\{ x \mid x \geq 0, \prod_{i=1}^n x_i = 0 \right\} \quad (1)$$

- D_2 is further subdivided into two regions

$$\begin{aligned} D'_2 &= \{ x \mid x \in D_2, (\lambda x) \in L_\Phi(u) \text{ for some } u > 0, \lambda > 0 \} \\ D''_2 &= \{ x \mid x \in D_2, (\lambda x) \notin L_\Phi(u) \text{ for some } u > 0, \lambda > 0 \} \end{aligned} \quad (2)$$

- The last segregation segregates D_2 into those points that are on a level set ($L_\Phi(u)$) and those points not on a level set.

- Looking at the definition if

$$x \in D_2 \Rightarrow x \geq 0, \prod_{i=1}^n x_i = 0 \quad (3)$$

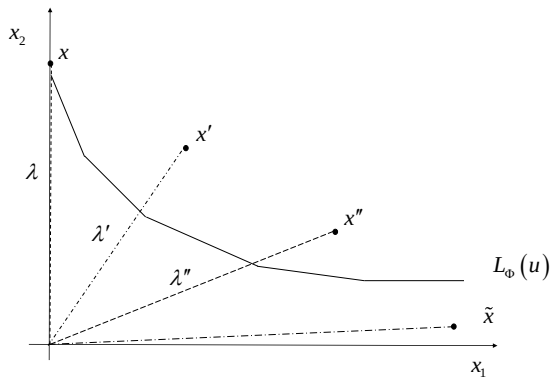
at least one x_i is equal to zero, but production is still possible (i.e., the input is not strongly necessary).

- Thus,

$$\lambda x \Rightarrow L_{\Phi}(u) \ni: u > 0 \quad (4)$$

- The second possibility is that inputs are strictly necessary.
- $D_2 = D'_2 \cup D''_2$
- Thus, the possible set of inputs D is defined as the union of all of these disjoint sets

$$D = \{0\} \cup D_1 \cup D'_2 \cup D''_2 \quad (5)$$



- The distance function ($\Psi(u, x)$) is then defined on D for the production possibilities sets

$$\Psi(u, x) = \begin{cases} \frac{\|x\|}{\|\zeta\|} & \text{for } x \in D_1 \cup D'_2, u > 0 \\ 0 & \text{for } x \in \{0\} \cup D''_2, u > 0 \\ +\infty & \text{for } x \in D, u = 0 \end{cases} \quad (6)$$

$$\zeta = \lambda_0 x$$

$$\lambda_0 = \min \{ \lambda \mid \lambda x \in L_\Phi(u) \}$$

- Breaking this definition down by parts, $\Psi(u, x)$ is a function of the output level u and an input vector x .
 - $L_\Phi(u)$ is the level-set of input that produce at least u output.
 - λ_0 is the minimum length along the ray that will generate that level of output (or an output level on that level set)

$$\lambda_0 x \in L_\Phi(u) \quad (7)$$

- Under the first scenario the input bundle gives a valid output level:
 - If $x \in D_1$ then the input bundle produces a valid output because it is an interior point in output space.
 - If $x \in D'_2$ then the input bundle is on one of the axis (i.e., $x_i = 0$ for some i), or one of the inputs is not strictly necessary.
- If the input bundle gives a valid output level the distance function is defined as

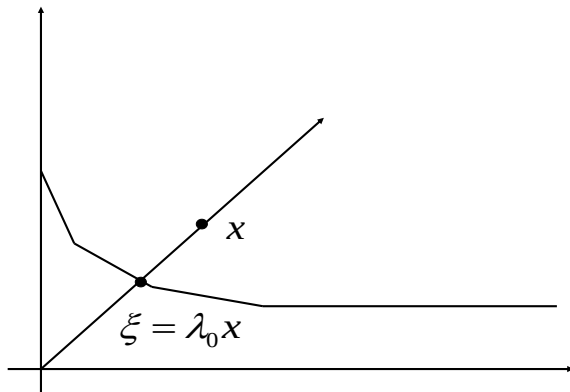
$$\Psi(u, x) = \frac{||x||}{||\zeta||} = \frac{||x||}{||\lambda_0 x||} \quad (8)$$

- Defining the norm of a vector

$$||x|| = \sqrt{\sum_{i=1}^n x_i^2} \quad (9)$$

- Thus

$$\Psi(u, x) = \frac{||x||}{\lambda_0 ||x||} = \frac{1}{\lambda_0} \quad (10)$$



- Infeasible Solutions

- If the original input vector does not yield a valid level set then

$$x \in \{0\} \cup D_2'', u > 0 \Rightarrow \Psi(u, x) = 0 \quad (11)$$

- Otherwise

$$x \in D, u = 0 \Rightarrow \Psi(u, x) = \infty \quad (12)$$

- Proposition 14: for any $u = [0, \infty)$

$$L_{\Phi}(u) = \{x \mid \Psi(u, x) \geq 1, x \in D\} \quad (13)$$

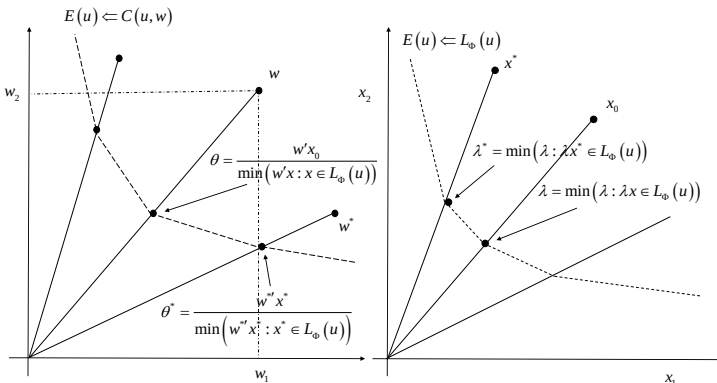
So the level set is defined as those input vectors x that have a distance function value of $\Psi(u, x) \geq 1$.

$$\begin{aligned} x \in L_{\Phi}(u), u > 0 &\Rightarrow \frac{\|x\|}{\|\zeta\|} \geq 1 \\ x \notin L_{\Phi}(u), x < \zeta &\Rightarrow \frac{\|x\|}{\|\zeta\|} < 1 \end{aligned} \quad (14)$$

Transcendental Data for Distance Function

Obs.	x_1	x_2	Y	Obs	x_1	x_2	Y
1	11.3426	9.4051	58.0610	26	3.3595	15.3676	38.9609
2	7.4441	9.5654	48.3455	27	13.0065	13.4051	68.5679
3	10.4883	5.1226	46.7750	28	5.3026	12.1365	44.5783
4	14.9439	8.0956	62.7818	29	9.3094	11.0515	55.7732
5	6.3922	15.3987	52.0170	30	5.4409	5.4613	35.5486
6	20.6834	15.3682	87.7413	31	5.4508	22.2421	53.9904
7	22.3023	11.9712	84.2381	32	9.9694	27.5997	75.3635
8	17.8273	12.3209	76.9367	33	6.5881	8.2770	43.8433
9	1.5203	12.0995	25.4184	34	9.3015	12.8227	58.2690
10	31.3959	5.4331	77.4007	35	11.9501	7.2292	54.9469
11	16.5561	17.7613	82.9715	36	7.3886	11.9547	51.4867
12	15.4640	7.9875	63.4893	37	10.2449	25.4174	74.4648
13	2.9278	15.7444	36.8916	38	7.5652	6.6255	43.6481
14	16.9118	20.3992	87.2536	39	5.9434	13.6374	48.5680
15	18.3845	10.6692	74.7291	40	7.4419	8.3519	46.4253
16	15.2791	33.5970	96.5524	41	19.4514	30.5316	104.5156
17	13.0784	14.4328	70.2588	42	12.7003	10.7296	63.5033
18	7.2867	38.0237	71.9314	43	25.4271	16.3276	97.8640
19	14.4622	13.2441	71.6269	44	7.0346	14.1833	52.9884
20	9.7641	21.3298	69.2198	45	15.2300	7.8365	62.7026
21	13.4874	8.7700	61.4279	46	17.5625	7.7100	66.4774
22	12.7641	18.4941	74.7924	47	29.6362	23.6347	116.7654
23	6.0977	20.4496	55.3839	48	10.1112	9.3295	55.0262
24	7.7591	4.1530	38.3968	49	21.6557	7.4715	72.2650
25	17.5761	14.5697	80.3519	50	4.1077	9.0759	36.4650

Obs	λ_i	x_1	x_2	Y	$\lambda_i x_1$	$\lambda_i x_2$
6	0.8089	20.6834	15.3682	84.9376	16.7317	12.4320
7	0.8547	22.3023	11.9712	83.7973	19.0624	10.2321
8	0.9662	17.8273	12.3209	75.9443	17.2242	11.9041
10	0.9582	31.3959	5.4331	77.0900	30.0835	5.2060
11	0.8725	16.5561	17.7613	80.5716	14.4448	15.4964
14	0.8151	16.9118	20.3992	85.7311	13.7841	16.6266
16	0.7103	15.2791	33.5970	94.6269	10.8523	23.8630
25	0.9111	17.5761	14.5697	78.9353	16.0142	13.2749
32	0.9935	9.9694	27.5997	74.1136	9.9045	27.4199
41	0.6380	19.4514	30.5316	102.7232	12.4098	19.4788
43	0.6977	25.4271	16.3276	95.8856	17.7395	11.3912
47	0.5488	29.6362	23.6347	116.2799	16.2641	12.9706



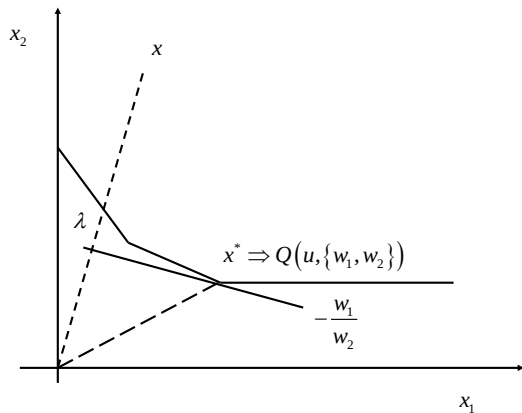
- Proposition 15: for any $u \in (0, +\infty)$, the isoquant of a production set $L_\Phi(u)$ consists of those input vectors $x \geq 0$ such that $\Psi(u, x) = 1$.
 - So the distance function can be used to define the isoquant.
- Proposition 17:

$$\Phi(x) = \max \{ u \mid \Psi(u, x) \geq 1 \} \quad x \in D \quad (15)$$

Thus, the distance function defines the production function.

- Following the general framework above for the distance function, the cost function is defined as

$$Q(u, p) = \min_x \{ p'x \mid x \in L_\Phi(u) \} \quad , p \in D, u \in [0, +\infty) \quad (16)$$



- From this definition

$$L_{\Phi}(u) = \{ x \mid p'x \geq Q(u, p) \} \quad \forall p \neq 0 \quad (17)$$

- Looking ahead, the two functions $Q(u, p)$ and $\Psi(u, x)$ are dualistically determined if

$$\begin{aligned} Q(u, p) &= \min_x \{ p'x \mid \Psi(u, x) \geq 1 \} \quad u \geq 0, p \geq 0 \\ \Psi(u, x) &= \inf_p \{ p'x \mid Q(u, p) \geq 1 \} \quad u \geq 0, x \geq 0 \end{aligned} \quad (18)$$