

An Application of Duality: Lecture XV

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Introduction and Setup

- Diewert, W. E. 1971. An Application of the Shephard Duality Theorem: A Generalized Leontief Production Function. *Journal of Political Economy* 79: 481-507.
- “The Shephard duality theorem states that technology may be equivalently represented by a production function, satisfying certain regularity conditions, or a cost function, satisfying certain regularity conditions.”
- “It is well known that, given fixed factor prices, and an n factor production function satisfying certain regularity conditions, we may derive a (minimum total) cost function under the assumption of minimizing behavior.”
- “What is not so well known is that, given a cost function satisfying certain regularity conditions, we may use this cost function to define a production function which in turn may be used to derive our original cost function.”

- “There are at least three ways of describing the technology of a single output, n inputs firm:
 - by means of a production function,
 - in terms of the firm’s production possibility sets $\dots$, and
 - by means of the firm’s cost cost function (if the firm purchases the services of factors at fixed prices). ”

Conditions on the Production Function $f(\cdot)$

- f is a real valued function of n real variables for every $x \geq 0$.
 f is finite if x is finite.
- $f(0) = 0$ and f is a nondecreasing function in x .
- $f(x^N)$ tends to plus infinity for at least one nonnegative sequence of vectors (x^N) .
- f is continuous from above or f is a right continuous function.
- f is quasiconcave over Ω .

Definitions

- Definition: A set X is convex if for every x_1 and x_2 that belongs to X and for every λ , $0 \leq \lambda \leq 1$, we have $\lambda x_1 + (1 - \lambda) x_2$ belongs to X .
- Definition: A real valued function f defined over a convex set X is concave if for every x_1 and x_2 belonging to X and $0 \leq \lambda \leq 1$, we have

$$f[\lambda x_1 + (1 - \lambda) x_2] \geq \lambda f(x_1) + (1 - \lambda) f(x_2). \quad (1)$$

- Definition: A real valued function f defined over a set X is quasiconcave if, for every real number y , the set $L(y) = [x : f(x) \geq y, x \text{ belongs to } X]$ is a convex set.

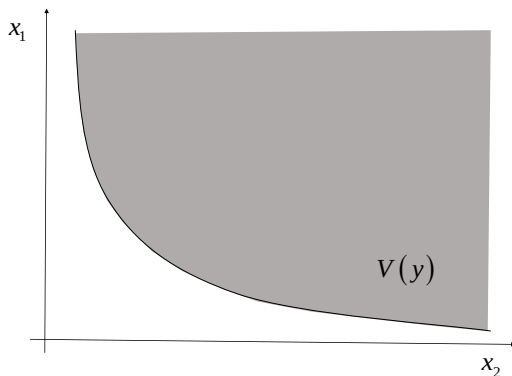
- Lemma: A real valued concave function defined over a convex set X is also quasiconcave.
- The proof is almost by definition. If x_1 and x_2 both belong to the level set $L(y)$, then

$$\begin{aligned} f[\lambda x_1 + (1 - \lambda) x_2] &\geq \lambda f(x_1) + (1 - \lambda) f(x_2) \\ &\geq \lambda y + (1 - \lambda) y \text{ by definition } x_1, x_2 \in L(y) \\ &= y \end{aligned} \quad (2)$$

- Definition: The production possibilities sets (or upper contour sets) are defined for every output level $y \geq 0$ by

$$L(y) = [x : f(x) \geq y, x \text{ nonnegative}] \quad (3)$$

Level Set



Conditions on the Production Possibilities Set $L(y)$

- $L(0) = \Omega$ for every $y > 0$, $L(y)$ is a nonempty closed set which does not contain the origin.
- For every $y > 0$, $L(y)$ is a convex set.
- If $x' \geq x$ (componentwise), where x belongs to $L(y)$, then x' also belongs to $L(y)$.
- If $y_1 \geq y_2$, then $L(y_1)$ is a subset of $L(y_2)$.
- For every x belonging to Ω , there exists a y such that x does not belong to $L(y)$.
- Graph L is a closed set where the graph

$$L = [x : f(x) \geq y, x \text{ belongs to } L(y), x \geq 0, y \geq 0] \quad (4)$$

- Theorem: The conditions on the production function imply that the conditions on the production possibility sets $L(y)$ are satisfied.
- Definition: Given a family of production possibilities sets $L(y)$ satisfying the conditions for the production possibility sets above define the following function on x

$$f(x) = \max_{x \geq 0} [\mu : x \text{ belongs to } L(y)] \quad (5)$$

- This definition actually goes backward – Assume that we define the production possibilities set $L(y)$, then we can define a function $f(x)$ as the maximum output that can be produced from any bundle of inputs as the highest production possibilities set that can be obtained from the set of inputs.

- **Condition a:** f is a real valued function for every $x \geq 0$, $f(x)$ is finite if x is finite.
 - The general idea is to show that the notion of a production possibility set (level set) implies a production function that is finite if x is finite.
 - To show that $f(x)$ is finite, we have to show that under the definition of $L(y)$ there must exist an x' such that $x' \geq x$ implies a higher production possibility set

$$f(x') = \max_{\mu \geq 0} [\mu : x \text{ belongs to } L(y')] \ni y' > y \quad (6)$$

- Looking backward, we see that for every x belonging to Ω there exists a y such that x does not belong to $L(y)$.
- Thus, for any x there exists a set $L(y)$ for $y \geq 0$ such that x does not belong to that set.
- Thus, x implies an $f(x)$ that is bounded (or an $f(x')$ exists such that $f(x') > f(x)$).

- **Condition b:** $f(0) = 0$ and f is nondecreasing in x .
 - The first section is based on the notion that $L(0) = \Omega$ and for every $y > 0$, $L(y)$ is a nonempty set.
 - The second part (nondecreasing) is based on the fact that if $y_1 \geq y_2$ then $L(y_1) \subset L(y_2)$.
 - By definition if $x \notin L(y_2)$ but $x_1 \in L(y_1)$, then $x_2 \in L(y_1)$ and $x \in L(y_2)$ such that $x_2 > x_1$.
- **Condition c:** $f(x^N)$ tends to plus infinity for at least one sequence of $x^N > 0$.
 - For simplicity let $y = N$. Thus, if we let $N \rightarrow \infty$, $y \rightarrow \infty$. Based on this we see

$$f(x^N) = \max [\mu : x^N \text{ belongs to } \mu \rightarrow y; y \rightarrow \infty] \quad (7)$$

which guarantees that for some sequence, the output values goes to infinity.

- **Condition d:** f is continuous from above, or f is a right continuous function.
 - Intuitively, if $L(y)$ is a closed function, then as you approach the production possibilities frontier from above, the production possibility set includes the production possibility frontier itself.

$$\begin{aligned}
 L^*(y) &= [x : f(x) \geq y, x \text{ nonnegative}] \\
 &= [x : \max \mu \geq y, x \text{ belongs to } L(\mu)] \text{ by definition of } f(x) \\
 &= [x : x \text{ belongs to } L(\mu) ; \mu \geq y] \\
 &= [x : x \text{ belongs to } L(y)] \text{ using the same results from the properties of } L(y) \\
 &= L(y) \text{ is a closed set}
 \end{aligned}
 \tag{8}$$

- **Condition e:** f is a quasiconcave function (see earlier lecture notes).

Properties of the Cost Function

- Definition: Given a family of production possibility sets $L(y)$ satisfying the conditions for the production possibility sets, then for any strictly positive price vector we may define the producers cost function $C(y; p)$ by

$$C(y; p) = \min_x [p'x : x \text{ belongs to } L(y)] \quad (9)$$

- $C(y; p)$ is a positive real valued function defined and finite for all finite $y > 0$ and strictly positive price vector.
- $C(y; p)$ is a nondecreasing left continuous function in y and tends to plus infinity as y tends to infinity for every strictly positive price vector.
- $C(y; p)$ is a nondecreasing function in p .
- $C(y; p)$ is (positive) linear homogeneous in p for every strictly positive price vector.
- $C(y; p)$ is a concave function in p for every $y \geq 0$.

- Definition: Define the family of sets $M(y)$, for $y = 0$ by

$$\begin{aligned} M(0) &= \Omega = (x : x \geq 0) \\ y > 0, M(y) &= [x : p'x \geq C(y; p) \text{ for every } p \gg 0, x \geq 0] \end{aligned} \quad (10)$$

- Theorem: Given a cost function satisfying the properties of the cost function discussed above, then the family of sets $M(y)$ generated by the cost function by means of the definitions above satisfy the condition for a production possibility set.

- Definition: Define the family of sets $M(y)$, for $y = 0$ by

$$\begin{aligned} M(0) &= \Omega = \{x : x \geq 0\} \\ y > 0, M(y) &= \{x : p'x \geq C(y; p) \text{ for every } p \gg 0, x \geq 0\} \end{aligned} \quad (11)$$

- Theorem: Given a cost function satisfying the properties of the cost function discussed above, then the family of sets $M(y)$ generated by the cost function by means of the definition above satisfy the conditions of the possibility set.

- **Condition a:** $L(0) = \Omega$ for every $y > 0$. $L(y)$ is a nonempty, closed set which does not contain the origin.
 - By definition $M(0) = \Omega$.
 - The trick is then to show that $M(y)$ is a nonempty set that does not contain the origin.
 - First to demonstrate that $M(y)$ is closed, Diewert uses the linear homogeneity of the cost function

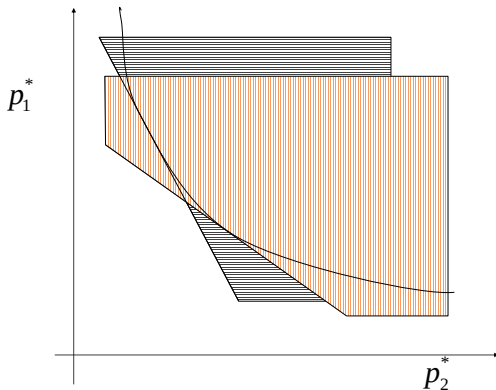
$$M(y) = [x : p'x \geq C(y; p) \text{ for every } p \gg 0]$$

$$\Rightarrow \left[x : p'x \geq C(y; p) \text{ for } p \gg 0 \ni \sum_{i=1}^N p_i = 1, x \geq 0 \right] \quad (12)$$

- Given that the cost function is homogeneous in prices, the price vector is invariant to normalization – thus, we can normalize it so that the sum of all prices is equal to one.

- Continued [**Condition a**]

- Thus, for any set of prices strictly greater than zero $[x : p'x = C(x, p), x \geq 0]$ is closed – given that any price vector can be so normalized, every possible price vector is closed.
- Given the intersection of a family sets is closed $M(y)$ is closed.
- Given any y , we know that for any arbitrary price vector, we can normalize the price vector so that its sum is one without changing the cost function due to linear homogeneity.
- Thus, we know that each price vector (set of price ratios) yields a closed subset.
- Next, the intersection of all such subsets that yields the same level of output, y , is then defined as $M(y)$ or the set of all price ratios that can be used to generate a given output set.



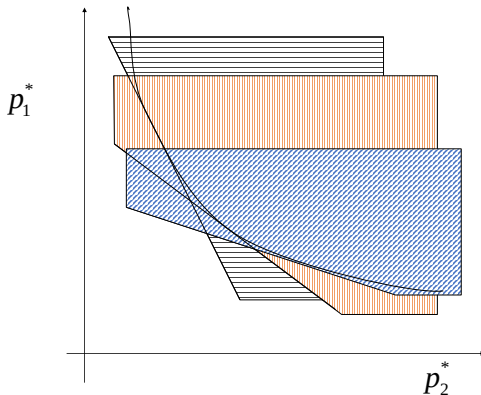
- Finally, $M(y)$ cannot be empty.
 - If $M(y)$ were empty then for any positive price vector we could normalize the output vector (set x to be a vector of ones, then $x^N = e \times N$ [$e = [1 \ 1 \ \cdots \ 1]$]).
 - Then there must exist a set of strictly positive prices p_N such that

$$p'_N x^N = N p'_N e = N < c(y; p_N) \quad (13)$$

but then $C(y; p)$ is unbounded for the price sequence (p_N) in violation of the property of the cost function.

- Condition b:** for every $y \geq 0$, $L(y)$ is a convex set.

Union of Sets



- **Condition c:** If $x' \geq x$ (componentwise), where x belongs to $L(y)$ then x' also belongs to $L(y)$.
 - This demonstration is rather straightforward. For a given price vector if $x' \geq x$ then $p'x' \geq p'x \geq C(y; p)$.
 - Since the cost function is nondecreasing in y , then if x belongs to $M(y)$ then x' belongs to $M(y)$.
- **Condition d:** If $y_1 \geq y_2$, then $L(y_1) \subset L(y_2)$.
 - By using the property of the cost function that increased output levels imply increased cost, this conclusion follows directly

$$y_1 \geq y_2 \Rightarrow C(y_1; p) \geq C(y_2; p) \quad (14)$$

Thus, by definition of the family of sets

$$\begin{aligned} M(y_1) &= [x : p'x \geq C(y_1; p) \text{ for every } p \gg 0, x \geq 0] \\ &\subset M(y_2) = [x : p'x \geq C(y_2; p) \text{ for every } p \gg 0, x \geq 0] \end{aligned} \quad (15)$$

- **Condition e:** For every x belonging to Ω , there exists a y such that x does not belong to $L(y)$.
 - This condition is demonstrated by the fact that for any x and price vector p strictly greater than zero, a y can be used to define a cost function $C(y; p)$ such that

$$p'x < C(y; p) \quad (16)$$

Since the cost function is defined for all y as y tends to infinity.

- **Condition f:** Graph L is a closed set.

Duality

- Taken together, the forgoing proofs prove the duality of the cost function.
 - Specifically, the properties of the production function imply the existence of production possibility sets (or input requirement sets).
 - The properties of the input requirement sets imply the existence of a cost function.
 - Going the other way, the properties of the cost function imply the existence of production possibility set which imply the existence of the production function.

$$\begin{aligned} f(x) &\Rightarrow \exists L(y) \Rightarrow \exists C(y; p) \\ C(y; p) &\Rightarrow \exists M(y) \{ = L(y) \} \Rightarrow \exists f(x) \end{aligned} \tag{17}$$

Shephard's Lemma

- Given this development, Diewert then proves Shephard's lemma

$$\frac{\partial C(y; p)}{\partial p_i} = x_i(y; p). \quad (18)$$

- Starting with the definition of the cost function, Diewert then hypothesizes a change in the price vector Δp where one price is changed

$$\begin{aligned} C(y; p + \Delta p) &= \min_x [x : x \text{ belongs to } M(y)] \\ &= (p + \Delta p)' x^* \\ &= (p + \Delta p)' (\bar{x} + \Delta x) \end{aligned} \quad (19)$$

- If we assume that x^* is the optimal choice of x at the new price vector, given the definition of optimal input use we have

$$\begin{aligned} C(y; p + \Delta p) &\leq (p + \Delta p)' \bar{x} \\ &= p' \bar{x} + \Delta p' \bar{x} \\ &= C(y; p) + \Delta p_i \bar{x}_i \Rightarrow \bar{x}_i \leq \frac{C(y; p + \Delta p) - C(y; p)}{\Delta p_i} \end{aligned} \quad (20)$$