

Comparative Statics and Duality of the Cost Function: Lecture XIV

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1 Comparative Statics

Comparative Statics

- Comparative statics with respect to changes in input prices.
- Most common results of the comparative statics with respect to input prices involve intuition about the derived demand functions.
 - From the primal approach we expect the demand functions for each input to be downward sloping with respect to input prices.
 - Starting from the cost function

$$\frac{\partial^2 c(w, y)}{\partial w_i \partial w_j} = \frac{\partial x_i^*(w, y)}{\partial w_j} \quad (1)$$

By Shephard's lemma.

- In addition, we know if $i = j$ then by the concavity of the cost function in prices

$$\frac{\partial^2 c(w, y)}{\partial w_i \partial w_i} = \frac{\partial x_i^*(w, y)}{\partial w_i} \leq 0 \quad (2)$$

- In addition, we know that by Young's theorem, the Hessian matrix for the cost function is symmetric

$$\frac{\partial^2 c(w, y)}{\partial w_i \partial w_j} = \frac{\partial^2 c(w, y)}{\partial w_j \partial w_i} \quad (3)$$

- The Hessian matrix for the cost function is also singular.
- Euler's Theorem: Euler's theorem is based on the definition of homogeneity

$$f(tx) = t^r f(x). \quad (4)$$

- Differentiating both sides with respect to t and applying the chain rule yields

$$\sum_{i=1}^N \frac{\partial f(tx)}{\partial (tx_i)} \frac{\partial (tx_i)}{\partial x_i} = r t^{r-1} f(x) \quad (5)$$

$$\sum_{i=1}^N \frac{\partial f(tx)}{\partial (tx_i)} x_i = r t^{r-1} f(x).$$

- Letting $t = 1$ then yields

$$\sum_{i=1}^N \frac{\partial f(x)}{\partial x_i} x_i = r f(x) \quad (6)$$

- Coupling this result with the observation that if a function is homogeneous of degree r , then its derivative is homogeneous of degree $r - 1$.
- We know that the input demand functions are homogeneous of degree zero in prices. Thus,

$$\sum_{j=1}^N \frac{\partial x_i^*(w, y)}{\partial w_j} w_j = \sum_{j=1}^N \frac{\partial^2 c(w, y)}{\partial w_i \partial w_j} w_j = 0 \quad (7)$$

- Multiplying this expression by $x_i^*(w, y)$ yields

$$\sum_{i=1}^N \frac{\partial x_i^*(w, y)}{\partial w_j} \frac{w_j}{x_i^*(w, y)} = \sum_{i=1}^N \epsilon_{ij} = 0 \quad (8)$$

- Given that we know that $\epsilon_{ii} < 0$, this result imposes restrictions on the cross-price elasticities.

- Briefly, let us prove the homogeneity of the marginal cost function. (It is a useful demonstration of the use of Shephard's lemma).
- Starting with the marginal cost, differentiate with respect to each price

$$\begin{aligned}\frac{\partial c(w, y)}{\partial y} &\Rightarrow \sum_{i=1}^N \frac{\partial^2 c(w, y)}{\partial w_i \partial y} w_i = \frac{\partial}{\partial y} \sum_{i=1}^N \frac{\partial c(w, y)}{\partial w_i} w_i \\ &= \frac{\partial}{\partial y} \sum_{i=1}^N x_i^*(w, y) w_i\end{aligned}\tag{9}$$

- The marginal cost function is homogeneous of degree one.

$$\begin{aligned} c(w, y) &= \sum_{i=1}^N x_i^*(w, y) w_i \Rightarrow \\ \sum_{i=1}^N \frac{\partial^2 c(w, y)}{\partial w_i \partial y} &= \frac{\partial}{\partial y} c(w, y) \end{aligned} \tag{10}$$

Comparative Statics with respect to Output Levels

- Following from the restrictions on the cross price elasticities above, the comparative statics with respect to output levels imply that not all inputs can be inferior or regressive.
- An inferior input is an input whose use declines as production increases while the use of a normal input increases as production increases.
 - Like the cross-price results above, we start with the sum of the differences of individual demand functions with respect to the level of output

$$\begin{aligned}\sum_{i=1}^N \frac{\partial x_i^*(w, y)}{\partial y} w_i &= \sum_{i=1}^N \frac{\partial^2 c(w, y)}{\partial w_i \partial y} w_i \\ &= \frac{\partial c(w, y)}{\partial y}\end{aligned}\tag{11}$$

- In order to develop the effect of output on total cost, we start with the original Lagrangian from the primal problem

$$L = w \times x + \lambda [y - f(x)] \Rightarrow \begin{cases} L_x = w_i - \lambda \frac{\partial f(x)}{\partial x_i} = 0 \\ L_\lambda = y - f(x) = 0 \end{cases} \quad (12)$$

- Solving for the output using the first-order condition on the Lagrange multiplier and differentiating the solution then yields

$$y = f(x) \Rightarrow dy = \sum_{i=1}^N \frac{\partial f(x)}{\partial x_i} dx_i \quad (13)$$

- From the first set of first-order conditions, we see

$$\frac{\partial f(x)}{\partial x_i} = \frac{w_i}{\lambda} \quad (14)$$

- Therefore

$$dy = \sum_{i=1}^N \frac{w_i}{\lambda} dx_i \Rightarrow \lambda dy = \sum_{i=1}^N w_i dx_i \Rightarrow \lambda = \sum_{i=1}^N w_i \frac{dx_i}{dy} \quad (15)$$

- Which proves the definition of λ consistent with the envelope theory.