

Definition and Properties of Cost Function: Lecture XIII

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From Previous Lectures

- In the preceding lectures we first developed the production function as a technological envelope demonstrating how inputs can be mapped into outputs.
- Next, we showed how these functions could be used to derive input demand, cost, and profit functions based on these functions and optimizing behavior.
- In this development, we stated that economist had little to say about the characteristics of the production function.
- We were only interested in these functions in the constraints that they imposed on optimizing behavior.

Insight of the Dual

- Thus, the insight added by the “dual” approach is the fact that we could simply work with the resulting optimizing behavior.
 - In some cases, this optimizing behavior can then be used to infer facts about the technology underlying it.
 - Gorman (1976) “Duality is about the choice of the independent variables in terms of which one defines a theory.”
 - Chambers (p.49) “The essence of the dual approach is that technology (or in the case of the consumer problem, preferences) constrains the optimizing behavior of individuals. One should therefore be able to use an accurate representation of optimizing behavior to study the technology.”

Cost Function Defined

- The cost function is defined as:

$$c(w, y) = \min_{x \geq 0} [w'x : x \in (y)] \quad (1)$$

- Literally, the cost function is the minimum cost of producing a given level of output from a specific set of inputs.
- This definition depends on the production set $V(y)$.
- In a specific instant such as the Cobb-Douglas production function we can define this production set analytically.
- Technology constrains the behavior of economic agents.
- For example, we will impose the restriction on the technology so that at least some input be used to produce any non-zero level of output.

- The goal is to place as few of restrictions on the behavior of economic agents as possible to allow for the derivation of a fairly general behavioral response.
- Not to loose sight of the goal, we are interested in be able to specify the cost function based on input prices and output prices:

$$c(w, y) = a_0 + \alpha'w + \frac{1}{2}w'Aw + \beta'y + \frac{1}{2}y'By + w'\Gamma y \quad (2)$$

- Is a standard form of the quadratic cost function that we use in empirical research.
 - We are interested in developing the properties under which this function represents optimizing behavior.

Shephard's Lemma

- In addition, we will demonstrate Shephard's lemma which states that

$$\frac{\partial c(w, y)}{\partial w_i} = x_i^*(w, y) = \alpha_i + A_{*i}w + \Gamma_{i*}y \quad (3)$$

- Or, that the derivative of the cost function with respect to the input price yields the demand equation for each input.

Properties of the Cost Function

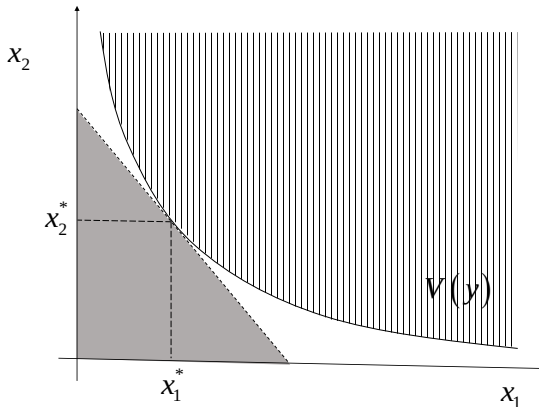
1. $c(w, y) > 0$ for $w > 0$ and $y > 0$ (nonnegativity).
2. If $w' \geq w$, then $c(w', y) \geq c(w, y)$ (nondecreasing in w).
3. The cost function is concave and continuous in w .
4. $c(tw, y) = tc(w, y), t > 0$ (the cost function is positively linear homogeneous).
5. If $y \geq y'$, then $c(w, y) \geq c(w, y')$ (nondecreasing in y); and
6. $c(w, 0) = 0$ (no fixed costs).

Level Set

- If the cost function is differentiable in w , then there exists a vector of costs minimizing the demand functions for each input formed from the gradient of the cost function with respect to w .
- In order to develop these properties, we begin with the basic notion that technology set is closed and nonempty.
- Thus $V(y)$ implies $x' \in V(y)$

$$\min_{x \geq 0} \{ w'x^* : w'(x^* - x) \leq 0; x \in V(y) \} \quad (4)$$

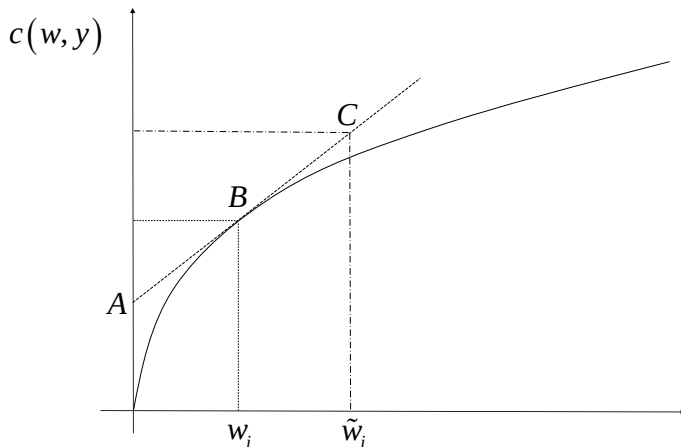
Graph of Level Set



Discussion of Properties

- **Property 2B.1** simply states that it is impossible to produce a positive output at zero cost. Going back to the production function, it was impossible to produce output without inputs.
 - Thus, given positive prices, it is impossible to produce outputs without a positive cost.
- **Property 2B.2** likewise seems obvious, if one of the input prices increases, then the cost of production increases.

Concavity



- First, if we constrain our discussion to the original input bundle, x_1 , it is clear that $w'_1 x_1 < w'_2 x_1$ if $w_2 > w_1$.
- Next, we have to establish that the change does not yield change in inputs such as the second price is lower than the first.
 - The conclusion follows from the previous equation

$$\min_{x \geq 0} \{ w'x : w'(x - \tilde{x}) \leq 0; x \in V(y) \} \quad (5)$$

- Taken together this result yields the *fundamental inequality of cost minimization*

$$(w_1 - w_2)'(x_1 - x_2) \leq 0 \quad (6)$$

If we focus on one price

$$(w_{1i} - w_{2i})(x_{1i} - x_{2i}) \leq 0 \quad (7)$$

- Continuous and concave in w . See Above Figure.
 - Note that A , B , and C lie on a straight line that is tangent to the cost function at B .
 - Movement from B to C would assume that the input bundle optimal at b is also optimal at C .
 - If, however, there are opportunities to substitute one input for another – these opportunities will be used if they produce a lower cost.
- To develop a more rigorous proof, let w_0 , w_1 , and w_w be vectors of prices, and x_1 and x_2 be associated input bundles such that

$$w_0 = \theta w_1 + (1 - \theta) w_2, \quad 0 \leq \theta \leq 1 \quad (8)$$

- Thus w_1 is on vector of inputs prices and w_2 is another vector of input prices. w_0 is then a linear combination of input prices.

- We then want to show that

$$c(w_0, y) \geq \theta c(w_1, y) + (1 - \theta) c(w_2, y) \quad (9)$$

- Let x_0 be the cost minimizing bundle associated with w_0 . By Cost minimization

$$\begin{aligned} w_1 x_0 &\geq w_1 x_1 \text{ and } w_2 x_0 \geq w_2 x_2 \\ c(w_0, y) &= w'_0 x_0 = [\theta w_1 + (1 - \theta) w_2]' x_0 \\ \theta w_1 x_0 + (1 - \theta) w_2 x_0 &\geq \theta c(w_1, y) + (1 - \theta) c(w_2, y) \quad (10) \\ &\Leftrightarrow \begin{cases} w'_1 x_0 \geq c(w_1, y) = w'_1 x_1 \\ w'_2 x_0 \geq c(w_2, y) = w'_2 x_2 \end{cases} \end{aligned}$$

- Positive linear homogeneity

$$\begin{aligned}
 c(tw, y) &= \min_{x \geq 0} \{ (tw)'x : x \in V(y) \} \\
 &= t \min_{x \geq 0} \{ w'x : x \in V(y) \} \\
 &= tc(w, y)
 \end{aligned} \tag{11}$$

- No fixed costs.

Shephard's Lemma

- In general, Shephard's lemma holds that

$$\nabla_w c(w, y) = \begin{bmatrix} \frac{\partial c(w, y)}{\partial w_1} \\ \frac{\partial c(w, y)}{\partial w_2} \\ \vdots \\ \frac{\partial c(w, y)}{\partial w_n} \end{bmatrix} = \begin{bmatrix} x_1^*(w, y) \\ x_2^*(w, y) \\ \vdots \\ x_n^*(w, y) \end{bmatrix} \quad (12)$$

- This proof is an application of the envelope theorem. First, assume that we want to maximize some general function

$$f(x_1, x_2, \dots, x_n, \alpha) \quad (13)$$

were we maximize $f(x, \alpha)$ through choosing x , but assume that α is fixed.

- To do this, we form the first-order conditions conditional on α

$$\begin{aligned} f_i(x_1, x_2, \dots, x_n, \alpha) = 0 &\Rightarrow x_i = x_i^*(\alpha) \\ y(x_1^*(\alpha), x_2^*(\alpha), \dots, x_n^*(\alpha), \alpha) &= \phi(\alpha) \end{aligned} \quad (14)$$

- The question is then: How does the solution change with respect to a change in α ?
- To see this we differentiate the optimum objective function value with respect to α to obtain

$$\frac{\partial y^*(.)}{\partial \alpha} = \frac{\partial \phi(\alpha)}{\partial \alpha} = \sum_{i=1}^n \frac{\partial f(.)}{\partial x_i} \frac{\partial x_i^*(\alpha)}{\partial \alpha} + \frac{\partial f(.)}{\partial \alpha}. \quad (15)$$

Given $\frac{\partial f(.)}{\partial x_i} = 0 \forall i$ this result implies that

$$\frac{\partial y^*(.)}{\partial \alpha} = \frac{\partial f(.)}{\partial \alpha}. \quad (16)$$

- Turning to the case of the constrained optimum

$$\begin{aligned} \max_{x_1, \dots, x_n} f(x_1, x_2, \dots, x_n, \alpha) \\ \text{s.t. } g(x_1, x_2, \dots, x_n, \alpha) = 0 \end{aligned} \quad (17)$$

Forming the Lagrangian

$$L = f + \lambda g \Rightarrow \left\{ \begin{array}{l} \frac{\partial L}{\partial x_i} = f_i + \lambda g_i = 0 \\ \frac{\partial L}{\partial \lambda} = g = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x_i = x_i^*(\alpha) \\ \lambda = \lambda^*(\alpha) \end{array} \right. . \quad (18)$$

$$y^*(x_1^*(\alpha), x_2^*(\alpha), \dots, x_n^*(\alpha), \alpha) \quad (19)$$

- Again, differentiating the optimum with respect to α , we get

$$\frac{\partial y^*(.)}{\partial \alpha} = \sum_{i=1}^n \frac{\partial f(.)}{\partial x_i} \frac{\partial x_i^*(\alpha)}{\partial \alpha} + \frac{\partial f(.)}{\partial \alpha} \quad (20)$$

but $\partial f(.)/\partial x_i \neq 0 \Leftrightarrow \partial f(.)/\partial x_i = \lambda \partial g(.)/\partial x_i$, or the changes are subject to the constraint.

- To work this out, we also have to differentiate the constraint with respect to α

$$\frac{\partial g(.)}{\partial \alpha} = \sum_{i=1}^n \frac{\partial g(.)}{\partial x_i} \frac{\partial x_i^*(\alpha)}{\partial \alpha} + \frac{\partial g(.)}{\partial \alpha} \quad (21)$$

- putting the two halves together (including the constraint) yields

$$\begin{aligned}\frac{\partial y^*(.)}{\partial \alpha} &= \sum_{i=1}^n \frac{\partial f(.)}{\partial x_i} \frac{\partial x_i^*(\alpha)}{\partial \alpha} \frac{\partial f(.)}{\partial \alpha} + \lambda \left(\sum_{i=1}^n \frac{\partial g(.)}{\partial x_i} \frac{\partial x_i^*(\alpha)}{\partial \alpha} + \frac{\partial g(.)}{\partial \alpha} \right) \\ \frac{\partial y^*(.)}{\partial \alpha} &= \sum_{i=1}^n \left(\frac{\partial f(.)}{\partial x_i} + \lambda \frac{\partial g(.)}{\partial x_i} \right) \frac{\partial x_i^*(\alpha)}{\partial \alpha} + \left(\frac{\partial f(.)}{\partial \alpha} + \lambda \frac{\partial g(.)}{\partial \alpha} \right)\end{aligned}\quad (22)$$

- Imposing the optimality result
($\partial f(.) / \partial x_i - \lambda \partial g(.) / \partial x_i = 0 \forall i$)

$$\frac{\partial y^*(.)}{\partial \alpha} = \frac{\partial f(.)}{\partial \alpha} + \lambda \frac{\partial g(.)}{\partial \alpha} \quad (23)$$

- Applying this formulation of the envelope theorem to the cost function, first we define the cost function as

$$\begin{aligned} c(w, y) &= \min_{x_1, x_2} w_1 x_1 + w_2 x_2 \\ \text{s.t. } &f(x_1, x_2) = y \end{aligned} \quad (24)$$

Formulating the Lagrangian

$$\begin{aligned} L &= w_1 x_1 + w_2 x_2 + \lambda (y_0 - f(x_1, x_2)) \\ \Rightarrow \frac{\partial c^*(w, y)}{\partial w_1} &= \frac{\partial L^*}{\partial w_1} = x_1^*(w, y) \end{aligned} \quad (25)$$

- More explicitly

$$\frac{\partial c^*(w, y)}{\partial w_1} = x_1^* + w_1 \frac{\partial x_1^*}{\partial w_1} + w_2 \frac{\partial x_2^*}{\partial w_1} \quad (26)$$

By the first-order conditions from Equation 25,

$$\left. \begin{aligned} w_1 &= \lambda \frac{\partial f}{\partial x_1} \\ w_2 &= \lambda \frac{\partial f}{\partial x_2} \end{aligned} \right\} \Rightarrow \frac{\partial c^*(w, y)}{\partial w_1} \quad (27)$$

$$= x_1^* + \lambda^* \left(\frac{f}{x_1} \frac{x_1^*}{\partial w_1} + \frac{\partial f}{\partial x_2} \frac{\partial x_2^*}{\partial w_1} \right)$$

- However, differentiating the constraint of the minimization problem we see

$$y_0 = f(x_1^*, x_2^*) \Rightarrow \frac{\partial y_0}{\partial w_1} = \frac{\partial f(\cdot)}{\partial x_1} \frac{\partial x_1^*}{\partial w_1} + \frac{\partial f(\cdot)}{\partial x_2} \frac{\partial x_2^*}{\partial w_2} = 0. \quad (28)$$

Thus, substituting the result from Equation 28 into Equation 27 yields

$$\frac{\partial c^*(w, y)}{\partial w_1} = x_1^*. \quad (29)$$