

Stochastic Error Function II – Maximum Likelihood: Lecture XI

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1 Normal–Half Normal

2 Maximum Likelihood

- Results

Normal-Half Normal

- Assumptions about errors

$$\begin{aligned}\nu_i &\sim N[0, \sigma_\nu^2] \\ u_i &\sim N^*[0, \sigma_u^2]\end{aligned}\tag{1}$$

ν_i and u_i independent ($N^*(.)$) denotes the half-normal.

- The normal distribution ν_i follows the standard normal (mean zero) formulation

$$f(\nu) = \frac{1}{\sqrt{2\pi}\sigma_\nu} \exp\left(-\frac{\nu^2}{2\sigma_\nu^2}\right)\tag{2}$$

- The half-normal distribution is represented by

$$g(u) = \frac{2}{\sqrt{2\pi}\sigma_u} \exp\left(-\frac{u^2}{2\sigma_u^2}\right)\tag{3}$$

- Assuming independence

$$f(u, \nu) = f(\nu) g(u) = \frac{2}{2\pi\sigma_\nu\sigma_u} \exp\left[-\frac{\nu^2}{2\sigma_\nu^2} - \frac{u^2}{2\sigma_u^2}\right] \quad (4)$$

- Since $\epsilon = \nu - u$, or by definition of the composed error term

$$f(u, \epsilon) = \frac{2}{2\pi\sigma_u\sigma_\nu} \exp\left[-\frac{u^2}{2\sigma_u^2} - \frac{(\epsilon + u)^2}{2\sigma_\nu^2}\right] \quad (5)$$

- Integrating out u , we obtain the marginal distribution function for ϵ – from Weinstein

$$\begin{aligned} x &= \frac{(X - \mu_X)}{\sigma_X} \\ y &= \frac{(Y - \mu_Y)}{\sigma_Y} \end{aligned} \quad (6)$$

x is distributed normal, while y is distributed half-normal

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \\ g(x) &= \begin{cases} \frac{f\left(\frac{y}{\sigma}\right)}{\sigma [1 - F(-a\sigma)]}, & y \geq -a \\ 0, & y < -a \end{cases} \end{aligned} \quad (7)$$

- Where

$$\sigma = \frac{\sigma_Y}{\sigma_X}$$

$$F(u) = \int_{-\infty}^u f(x) dx \quad (8)$$

- Note if $a = 0$

$$g(y) \propto \frac{f\left(\frac{y}{\sigma}\right)}{\sigma [1 - F(0)]}$$

$$= \frac{f\left(\frac{y}{\sigma}\right)}{\sigma [1 - 0.5]} \quad (9)$$

$$= \frac{2f\left(\frac{y}{\sigma}\right)}{\sigma} \Rightarrow \frac{2}{\sqrt{2\pi}\sigma_X\sigma} \exp\left(-\frac{y^2}{(\sigma\sigma_X)^2}\right)$$

- Substuting in

$$\sigma = \frac{\sigma_Y}{\sigma_X}$$

$$g(y) = \frac{2}{\sqrt{2\pi}\sigma_X \frac{\sigma_Y}{\sigma_X}} \exp\left(-\frac{y^2}{\left(\frac{\sigma_Y}{\sigma_X}\sigma_X\right)^2}\right) \quad (10)$$

$$\frac{2}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{y^2}{\sigma_Y^2}\right)$$

- By integration

$$\begin{aligned}
 Q(t) &= \int_{-\infty}^t \int_{-\infty}^{\infty} g(z-x) f(x) dx dz \\
 &= \frac{1}{\sigma \left[1 - \Phi\left(-\frac{a}{\sigma}\right)\right]} \int_{-\infty}^t \int_{-\infty}^{z+a} f\left(\frac{z-x}{\sigma}\right) f(x) dx dz \\
 &= \frac{1}{\left[1 - \Phi\left(-\frac{a}{\sigma}\right)\right]} \int_{-\infty}^{t+a} \left[\Phi\left(\frac{t-x}{\sigma}\right) - \Phi\left(\frac{-a}{\sigma}\right)\right] f(x) dx
 \end{aligned} \tag{11}$$

- Note that if $a = 0$

$$\begin{aligned}
 Q(t) &= \frac{1}{1 - \Phi(0)} \int_{-\infty}^t \left[\Phi\left(\frac{t-x}{\sigma}\right) - \Phi(0) \right] f(x) dx \\
 &= 2 \int_{-\infty}^t \left[\Phi\left(\frac{t-x}{\sigma}\right) - 0.5 \right] f(x) dx \\
 &= 2 \int_{-\infty}^t \frac{1}{2} \left[2\Phi\left(\frac{t-x}{\sigma}\right) - 1 \right] f(x) dx
 \end{aligned} \tag{12}$$

- Completing the derivation

$$\begin{aligned}
 f(\epsilon) &= \int_0^\infty f(u, \epsilon) du \\
 &= \frac{2}{\sqrt{2\pi}\sigma} \left[1 - \Phi\left(\frac{\epsilon\lambda}{\sigma}\right) \right] \exp\left[-\frac{\epsilon^2}{2\sigma^2}\right] \\
 &= \frac{2}{\sigma} \phi\left(\frac{\epsilon}{\sigma}\right) \Phi\left(-\frac{\epsilon\lambda}{\sigma}\right) \\
 \sigma &= \sqrt{\sigma_u^2 + \sigma_v^2}, \quad \lambda = \frac{\sigma_u}{\sigma_v}
 \end{aligned} \tag{13}$$

- Note that as $\lambda \rightarrow 0$ either $\sigma_v^2 \rightarrow \infty$ or $\sigma_u^2 \rightarrow 0$ or the symmetric error dominates the one-sided component.
- Note that as $\lambda \rightarrow \infty$ either $\sigma_u^2 \rightarrow \infty$ or $\sigma_v^2 \rightarrow 0$ or the symmetric component dominates the one-sided component.

Maximum Likelihood

- The parameters of the model can be estimated by maximizing

$$\ln(L) \propto N \ln(\sigma) + \sum_{i=1}^N \ln \left(\Phi \left(-\frac{\epsilon_i \lambda}{\sigma} \right) \right) - \frac{1}{2\sigma^2} \sum_{i=1}^N \epsilon_i^2 \quad (14)$$

Results

Parameter	Estimate
A_0	4.99564*** (0.035674)
A_1	0.00903 (0.00706)
A_2	0.00504 (0.00500)
A_3	0.00452 (0.00424)
σ	0.45639*** (0.02126)
λ	5.08765*** (0.77545)

More Mess

$$\lambda = \frac{\sigma_u}{\sigma_v} \Rightarrow \sigma_u = \lambda \sigma_v$$
$$\therefore \sigma_u^2 = \lambda^2 \sigma_v^2$$
(15)

$$\sigma^2 = \lambda^2 \sigma_v^2 + \sigma_v^2$$
$$= \sigma_v^2 (\lambda^2 + 1)$$
$$\therefore \sigma_v^2 = \frac{\sigma^2}{\lambda^2 + 1}$$
$$\sigma_u^2 = \sigma^2 - \sigma_v^2$$
(16)

$$\sigma_v = 4.98, \sigma_u = 1.04$$
(17)

Comparing Results

Parameter	Stochastic Frontier	Ordinary Least Squares
A_0	4.99564 (0.03574)	4.58582 (0.05570)
A_1	0.00903 (0.00706)	0.01265 (0.01171)
A_2	0.00504 (0.00500)	0.01677 (0.00728)
A_3	0.00452 (0.00424)	0.01322 (0.00625)