

Some Simple Production Mechanics: Lecture IV

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Basic Lagrangian

- Mathematically, the problem becomes

$$\begin{aligned} \max_x f(x) \\ \text{s.t. } g(x) = b \end{aligned} \tag{1}$$

- Transforming the problem into the Lagrange form yields

$$\max_{x,\lambda} L = f(x) - \lambda (b - g(x)) \tag{2}$$

where L is the value of the constrained optimum and λ is the Lagrange multiplier which depicts the derivative of the objective function with respect to the right-hand side of the constraint at the optimal point.

First-Order Conditions

- The first-order necessary conditions for the optimum are then derived as

$$\nabla_{(\lambda,x)} L = \begin{bmatrix} b - g(x) \\ -\lambda \frac{\partial g(x)}{\partial x_1} + \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ -\lambda \frac{\partial g(x)}{\partial x_n} + \frac{\partial f(x)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3)$$

which are the standard first-order conditions.

Second-Order Conditions

- Deriving the Bordered Hessian

$$\nabla^2_{(\lambda,x)(\lambda,x)} L = \begin{bmatrix} 0 & \frac{\partial g(x)}{\partial x_1} & \cdots & \frac{\partial g(x)}{\partial x_n} \\ \frac{\partial g(x)}{\partial x_1} & -\lambda \frac{\partial^2 g(x)}{\partial x_1^2} + \frac{\partial^2 f(x)}{\partial x_1^2} & \cdots & -\lambda \frac{\partial^2 g(x)}{\partial x_1 \partial x_n} + \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g(x)}{\partial x_n} & -\lambda \frac{\partial^2 g(x)}{\partial x_n \partial x_1} + \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & -\lambda \frac{\partial^2 g(x)}{\partial x_n^2} + \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix} \quad (4)$$

Bordered Hessian

• Or

$$\begin{bmatrix} d\lambda \\ dx_1 \\ \vdots \\ dx_n \end{bmatrix}' \begin{bmatrix} 0 & \frac{\partial g(x)}{\partial x_1} & \dots & \frac{\partial g(x)}{\partial x_n} \\ \frac{\partial g(x)}{\partial x_1} & -\lambda \frac{\partial^2 g(x)}{\partial x_1^2} + \frac{\partial^2 f(x)}{\partial x_1^2} & \dots & -\lambda \frac{\partial^2 g(x)}{\partial x_1 \partial x_n} + \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g(x)}{\partial x_n} & -\lambda \frac{\partial^2 g(x)}{\partial x_n \partial x_1} + \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \dots & -\lambda \frac{\partial^2 g(x)}{\partial x_n^2} + \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ \vdots \\ dx_n \end{bmatrix} \quad (5)$$

for all $d\lambda, dx_1, \dots, dx_n$.

Simpler Form

- Based on this structure, we consider a slightly different form of the Hessian matrix

$$\begin{bmatrix} dY \\ dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}' \begin{bmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} dY \\ dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \quad (6)$$

- Working through this matrix multiplication yields

$$2 \, dY \left(\sum_{i=1}^3 f_i x_i \right) + \sum_{i=1}^3 \sum_{j=1}^3 f_{ij} dx_i dx_j \quad (7)$$

- This first part of the expression guarantees that dx_i/dx_j is constrained so that there is no change in output (or the movement is around an isoquant)

$$dY = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 = 0 \quad (8)$$

- To refine the measure, let us restrict our attention to a production function with three inputs

$$\begin{bmatrix} dY \\ dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}' \begin{bmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} dY \\ dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \leq 0 \quad (9)$$

- Next, consider using the linear relationship in Equation 9 to solve for the value of a change in the value of x_2 (dx_2)

$$\begin{bmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} dY \\ dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ dx_2 \\ 0 \end{bmatrix}. \quad (10)$$

- Thus, we want to solve for the change in the other variables based on a change in x_2 that keeps the output constant (e.g., such that the production constraint is met). Simplifying this expression slightly

$$\begin{bmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} \frac{dY}{dx_2} \\ \frac{dx_1}{dx_2} \\ \frac{dx_2}{dx_2} \\ \frac{dx_3}{dx_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \quad (11)$$

Cramer's Rule

$$\frac{dx_1}{dx_2} = \frac{\begin{vmatrix} 0 & 0 & f_2 & f_3 \\ f_1 & 0 & f_{12} & f_{13} \\ f_2 & 1 & f_{22} & f_{23} \\ f_3 & 0 & f_{32} & f_{33} \end{vmatrix}}{\begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{vmatrix}} = \frac{(-1)^{(1+2)} \begin{vmatrix} 0 & f_2 & f_3 \\ f_1 & f_{12} & f_{13} \\ f_3 & f_{32} & f_{33} \end{vmatrix}}{\begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{vmatrix}} \quad (12)$$

Single Product Primal Optimization

- Profit Maximization

$$\begin{aligned} \max_{x_1, x_2} \pi &= p_Y x_1^\alpha x_2^\beta - w_1 x_1 - w_2 x_2 \\ \left. \begin{aligned} \frac{\partial \pi}{\partial x_1} &= p_Y \alpha \frac{Y}{x_1} - w_1 = 0 \\ \frac{\partial \pi}{\partial x_2} &= p_Y \beta \frac{Y}{x_2} - w_2 = 0 \end{aligned} \right\} &\Rightarrow x_2 = \frac{w_1}{w_2} \frac{\beta}{\alpha} x_1 \end{aligned} \quad (13)$$

$$p_Y \alpha \frac{Y}{x_1} - w_1 = 0 \Rightarrow p_Y x_1^{\alpha-1} x_2^\beta - w_1 = 0$$

$$p_Y \alpha \left[\frac{w_1}{w_2} \frac{\beta}{\alpha} x_1 \right]^\beta = w_1$$

$$p_Y \alpha \left[\frac{w_1}{w_2} \right]^\beta \left[\frac{\beta}{\alpha} \right]^\beta x_1^{\alpha+\beta-1} = w_1 \quad (14)$$

$$p_Y \left[\frac{\beta}{w_2} \right]^\beta \left[\frac{w_1}{\alpha} \right]^{\beta-1} = x_1^{1-\alpha-\beta}$$

$$\Rightarrow x_1^* (p_Y, w_1, w_2) = p_Y^{\frac{1}{1-\alpha-\beta}} \left[\frac{w_1}{\alpha} \right]^{\frac{1-\beta}{1-\alpha-\beta}} \left[\frac{\beta}{w_2} \right]^{\frac{\beta}{1-\alpha-\beta}}$$

$$\begin{aligned}
 x_2 &= \frac{w_1}{w_2} \frac{\beta}{\alpha} \left[p_Y^{\frac{1}{1-\alpha-\beta}} \left(\frac{w_1}{\alpha} \right)^{\frac{\beta-1}{1-\alpha-\beta}} \left(\frac{\beta}{w_2} \right)^{\frac{\beta}{1-\alpha-\beta}} \right] \\
 &= p_Y^{\frac{1}{1-\alpha-\beta}} \left[\frac{w_1}{\alpha} \right]^{1+\frac{\beta-1}{1-\alpha-\beta}} \left[\frac{\beta}{w_2} \right]^{1+\frac{\beta}{1-\alpha-\beta}} \\
 1 + \frac{\beta-1}{1-\alpha-\beta} &= \frac{1-\alpha-\beta+\beta-1}{1-\alpha-\beta} = \frac{-\alpha}{1-\alpha-\beta} \\
 1 + \frac{\beta}{1-\alpha-\beta} &= \frac{1-\alpha-\beta+\beta}{1-\alpha-\beta} = \frac{1-\alpha}{1-\alpha-\beta} \\
 \Rightarrow x_2^*(p_Y, w_1, w_2) &= p_Y^{\frac{1}{1-\alpha-\beta}} \left[\frac{\alpha}{w_1} \right]^{\frac{\alpha}{1-\alpha-\beta}} \left[\frac{w_2}{\beta} \right]^{\frac{\alpha-1}{1-\alpha-\beta}}
 \end{aligned} \tag{15}$$

- By convention

$$\begin{aligned}x_1^*(p_Y, w_1, w_2) &= p_Y^{\frac{1}{1-\alpha-\beta}} \left[\frac{\alpha}{w_1} \right]^{\frac{1-\beta}{1-\alpha-\beta}} \left[\frac{\beta}{w_2} \right]^{\frac{\beta}{1-\alpha-\beta}} \\x_2^*(p_Y, w_1, w_2) &= p_Y^{\frac{1}{1-\alpha-\beta}} \left[\frac{\beta}{w_2} \right]^{\frac{1-\alpha}{1-\alpha-\beta}} \left[\frac{\alpha}{w_1} \right]^{\frac{\alpha}{1-\alpha-\beta}}\end{aligned}\tag{16}$$

Supply Curve

$$\begin{aligned} Y^*(p_Y, w_1, w_2) &= \left[p_Y^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{w_1} \right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{\beta}{w_2} \right)^{\frac{\beta}{1-\alpha-\beta}} \right]^{\alpha} \\ &\times \left[p_Y^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{w_1} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{\beta}{w_2} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} \right]^{\beta} \\ &= p_Y^{\frac{\alpha+\beta}{1-\alpha-\beta}} \left[\frac{\alpha}{w_1} \right]^{\frac{\alpha-\alpha\beta+\alpha\beta}{1-\alpha-\beta}} \left[\frac{\beta}{w_2} \right]^{\frac{\beta-\alpha\beta+\alpha\beta}{1-\alpha-\beta}} \\ &= p_Y^{\frac{\alpha+\beta}{1-\alpha-\beta}} \left[\frac{\alpha}{w_1} \right]^{\frac{\alpha}{1-\alpha-\beta}} \left[\frac{\beta}{w_2} \right]^{\frac{\beta}{1-\alpha-\beta}} \end{aligned} \tag{17}$$

Profit Function

$$\begin{aligned}\pi^*(p_Y, w_1, w_2) = & p_Y \left[p_Y^{\frac{\alpha+\beta}{1-\alpha-\beta}} \left(\frac{\alpha}{w_1} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{\beta}{w_2} \right)^{\frac{\beta}{1-\alpha-\beta}} \right] \\ & - w_1 \left[p_Y^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{w_1} \right)^{\frac{1-\beta}{1-\alpha-\beta}} \left(\frac{\beta}{w_2} \right)^{\frac{\beta}{1-\alpha-\beta}} \right] \\ & - w_2 \left[p_Y^{\frac{1}{1-\alpha-\beta}} \left(\frac{\alpha}{w_1} \right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{\beta}{w_2} \right)^{\frac{1-\alpha}{1-\alpha-\beta}} \right]\end{aligned}\quad (18)$$

Cost Function

- Cost Minimization

$$\begin{aligned} \min_{x_1, x_2} w_1 x_1 + w_2 x_2 \\ \text{s.t. } x_1^\alpha x_2^\beta = Y \end{aligned} \quad (19)$$

- Lagrangian

$$\begin{aligned} L &= w_1 x_1 + w_2 x_2 + \lambda (Y - x_1^\alpha x_2^\beta) \\ \frac{\partial L}{\partial x_1} &= w_1 - \lambda \alpha x_1^{\alpha-1} x_2^\beta = 0 \Rightarrow w_1 - \lambda \alpha \frac{Y}{x_1} = 0 \\ \frac{\partial L}{\partial x_2} &= w_2 - \lambda \beta x_1^\alpha x_2^{\beta-1} = 0 \Rightarrow w_2 - \lambda \beta \frac{Y}{x_2} = 0 \\ \frac{\partial L}{\partial \lambda} &= Y - x_1^\alpha x_2^\beta = 0 \end{aligned} \quad (20)$$

- Taking the ratio of the first two first-order conditions yields

$$\begin{aligned}\frac{\lambda \alpha \frac{Y}{x_1}}{\lambda \beta \frac{Y}{x_2}} &= \frac{w_1}{w_2} \Rightarrow \frac{\alpha x_2}{\beta x_1} = \frac{w_1}{w_2} \\ \Rightarrow x_2 &= \frac{w_1 \beta}{w_2 \alpha} x_1 \\ \Rightarrow Y - x_1^\alpha \left[\frac{w_1 \beta}{w_2 \alpha} x_1 \right]^\beta &= 0 \\ \Rightarrow Y - x_1^{\alpha+\beta} \left[\frac{w_1 \beta}{w_2 \alpha} \right]^\beta &= 0\end{aligned}\tag{21}$$

$$\Rightarrow \begin{cases} x_1^*(Y, w_1, w_2) = Y^{\frac{1}{\alpha+\beta}} \left[\frac{w_2 \alpha}{w_1 \beta} \right]^{\frac{\beta}{\alpha+\beta}} \\ x_2^*(Y, w_1, w_2) = Y^{\frac{1}{\alpha+\beta}} \left[\frac{w_1 \beta}{w_2 \alpha} \right]^{\frac{\alpha}{\alpha+\beta}} \end{cases} \quad (22)$$

- Substituting these results into the cost function yields

$$C(Y, w_1, w_2) = w_1 Y^{\frac{1}{\alpha+\beta}} \left[\frac{w_2 \alpha}{w_2 \beta} \right]^{\frac{\beta}{1-\alpha-\beta}} + w_2 Y^{\frac{1}{\alpha+\beta}} \left[\frac{w_1 \beta}{w_2 \alpha} \right]^{\frac{\alpha}{\alpha+\beta}} \quad (23)$$

- Note that the profit function can be derived from the cost function

$$\begin{aligned}\max_Y \pi (P_y, w_1, w_2) &= p_Y Y - C (Y, w_1, w_2) \\ Y^* (p_Y, w_1, w_2) &\Rightarrow Y \ni: p_Y - \frac{\partial C (Y, w_1, w_2)}{\partial Y} = 0\end{aligned}\tag{24}$$

Multiproduct Primal Functions

- As discussed in class, the production function can be extended to multiple outputs. However, closed form functions are somewhat limited.
- In this section, I want to briefly discuss the theoretical application of the multiproduct primal function within the context of a planting problem.
- Specifically, assume that there exists a multivariate production function, $f(y,x)$, where y is a vector of (two) outputs, and x is a vector of (two) inputs.
- The profit function for this formulation can be formulated as

$$\max_{y_1, y_2, x_1, x_2} p_1 y_1 + p_2 y_2 - w_1 x_1 - w_2 x_2$$
$$f(y, x) = 0 \tag{25}$$

$$L = p_1 y_1 + p_2 y_2 - w_1 x_1 - w_2 x_2 - \lambda (f(y, x))$$

$$\frac{\partial L}{\partial y_1} = p_1 - \lambda \frac{\partial f(.)}{\partial y_1} \leq 0 ; \frac{\partial L}{\partial y_2} = p_2 - \lambda \frac{\partial f(.)}{\partial y_2}$$

$$\frac{\partial L}{\partial y_1} y_1 = 0 ; \frac{\partial L}{\partial y_2} y_2 = 0 \quad (26)$$

$$\frac{\partial L}{\partial x_1} = -w_1 - \lambda \frac{\partial f(.)}{\partial x_1} \leq 0 ; \frac{\partial f(.)}{\partial x_2} = -w_2 - \lambda \frac{\partial f(.)}{\partial x_2} \leq 0$$

$$\frac{\partial L}{\partial x_1} x_1 = 0 ; \frac{\partial L}{\partial x_2} x_2 = 0$$

- Taken together, these conditions imply that the value of marginal product of each input equals the input price, if positive quantities of each output are produced and positive quantities of inputs are used.
- These conditions admit three possible solutions.
 - First, if only y_1 is produced

$$\begin{aligned}\frac{\partial L}{\partial y_1} &= p_1 - \lambda \frac{\partial f(.)}{\partial y_1} = 0, y_1 > 0 \\ \frac{\partial L}{\partial y_2} &= p_2 - \lambda \frac{\partial f(.)}{\partial y_2} < 0, y_2 = 0\end{aligned}\tag{27}$$

- First order conditions

- Second, only y_2 could be produced

$$\begin{aligned}\frac{\partial L}{\partial y_1} &= p_1 - \lambda \frac{\partial f(.)}{\partial y_1} < 0, y_1 = 0 \\ \frac{\partial L}{\partial y_2} &= p_2 - \lambda \frac{\partial f(.)}{\partial y_2} = 0, y_2 > 0\end{aligned}\tag{28}$$

- Third, both outputs could be produced

$$\begin{aligned}\frac{\partial L}{\partial y_1} &= p_1 - \lambda \frac{\partial f(.)}{\partial y_1} = 0, y_1 > 0 \\ \frac{\partial L}{\partial y_2} &= p_2 - \lambda \frac{\partial f(.)}{\partial y_2} = 0, y_2 > 0\end{aligned}\tag{29}$$

$$\frac{p_1}{p_2} = \frac{\lambda \frac{\partial f(.)}{\partial y_1}}{\lambda \frac{\partial f(.)}{\partial y_2}} = \frac{\partial y_2}{y_1} \quad (30)$$

