

Substituting One Input for Another in Production: Lecture III

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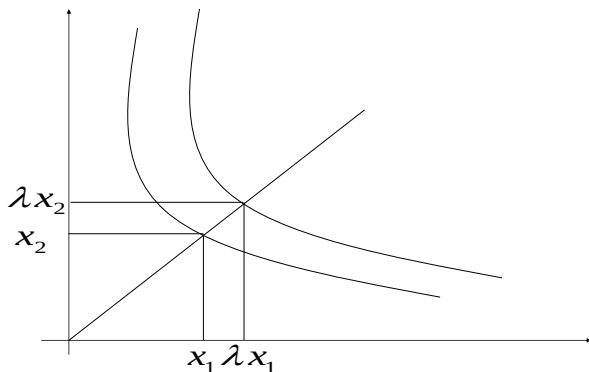
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Elasticity of Scale and Law of Variable Proportions

- We continue to develop the economies of production through a ray from the origin.
- Last lecture, we developed the notion of returns to scale by looking at changes in production along a ray from the origin.

Ray Depiction of Technology



Ray Formulation of Elasticity of Scale

- Following the definition of this ray, we defined the elasticity of scale

$$\epsilon = \left. \frac{\partial \ln [f(\lambda x)]}{\partial \ln(\lambda)} \right|_{\lambda=1} \quad (1)$$

$$\left. \frac{\partial \ln [f(\lambda x)]}{\partial \ln(\lambda)} \right|_{\lambda=1} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{x_i}{y} = \sum_{i=1}^n \epsilon_i$$

- Building on these definitions, we next define the *ray average product* as

$$RAP = \frac{f(\lambda x)}{\lambda} \quad (2)$$

where λ is a strictly positive scalar.

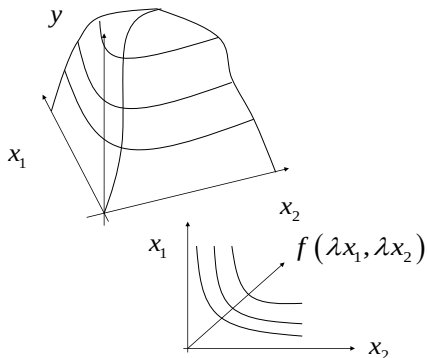
Ray Marginal Product

- In addition, we define the *ray marginal product*

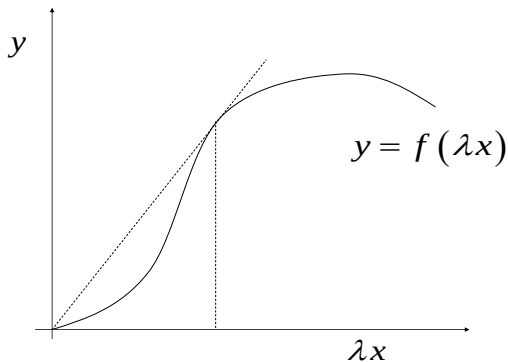
$$\begin{aligned}\lambda x \rightarrow z(x) &\Rightarrow \frac{\partial f(\lambda x)}{\partial \lambda} = \frac{\partial f(z(x))}{\partial \lambda} \\ &= \frac{\partial f(z(x))}{\partial z(x)} \frac{\partial z(x)}{\partial \lambda}\end{aligned}\quad (3)$$

$$\begin{aligned}\sum_{i=1}^n \frac{\partial f(\lambda x)}{\partial x_i} x_i \\ \frac{\partial^2 f(\lambda x)}{\partial \lambda^2} = \sum_{j=1}^n \sum_{i=1}^n \frac{\partial f}{\partial x_i \partial x_j} x_i x_j\end{aligned}\quad (4)$$

Production Frontier



Average Product/Marginal Product



- Differentiating the *ray average product* yields

$$\begin{aligned}
 \frac{\partial (RAP)}{\partial \lambda} &= \frac{\partial \frac{f(\lambda x)}{\lambda}}{\partial \lambda} \frac{1}{\lambda} \\
 &= \frac{\partial f(\lambda x)}{\partial \lambda} \frac{1}{\lambda} + f(\lambda x) \frac{\partial \frac{1}{\lambda}}{\partial \lambda} \\
 &= \frac{1}{\lambda} \left[\frac{\partial f(\lambda f(\lambda x))}{\partial \lambda} - \frac{f(\lambda x)}{\lambda} \right] \\
 &= \frac{1}{\lambda} [RMP - RAP]
 \end{aligned} \tag{5}$$

- Which states that the *ray average product* is maximum when it is equal to the *ray marginal product* (RMP).
- Note that this relationship is the same as the univariate relationship

$$\frac{d APP}{d x} = \frac{1}{x} (MPP - APP) \quad (6)$$

- Applying these relationships to the Zellner production function

$$\phi(x_1, x_2) = \frac{ax_1^3}{\exp\left[b\frac{x_1}{x_2}\right] - 1} \quad (7)$$

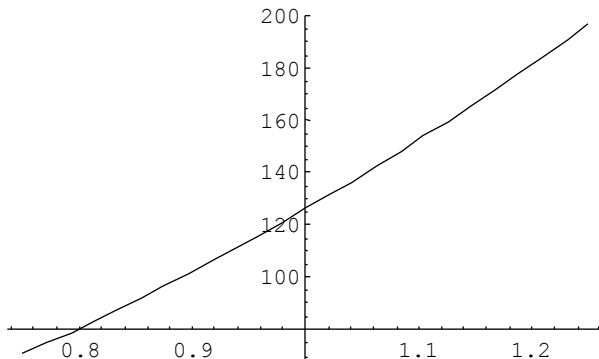
Note that by definition

$$\phi(\lambda x_1, \lambda x_2) = \frac{a\lambda^3 x_1^3}{\exp\left[b\frac{\lambda x_1}{\lambda x_2}\right] - 1} = \frac{a\lambda^3 x_1^3}{\exp\left[b\frac{x_1}{x_2}\right] - 1} \quad (8)$$

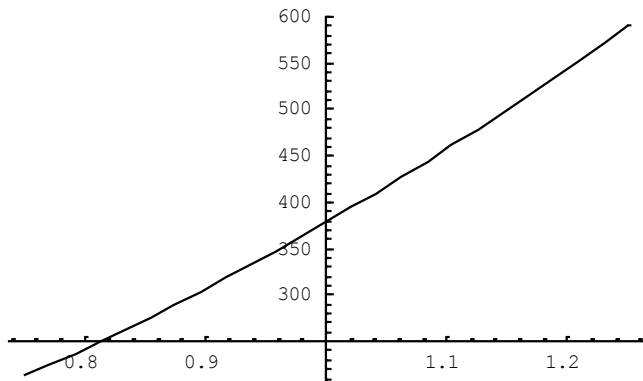
- Thus, x_2 does not affect the scale economies.
- The RMP is then

$$RMP = \frac{3a\lambda^3 x_1^3}{\exp\left[b\frac{x_1}{x_2}\right] - 1} \quad (9)$$

Ray Average Product

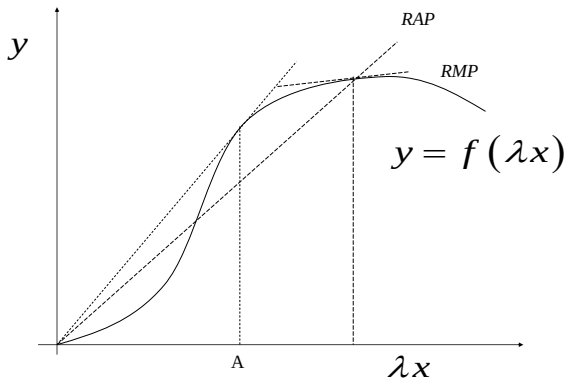


Ray Marginal Product



- The results also indicate that the *ray average product* reaches a maximum at

$$\frac{\partial \ln [f(\lambda x)]}{\partial \ln (\lambda)} = 1 \quad (10)$$



- If $\lambda = 1$ at point A , the production function exhibits constant returns to scale at x , since

$$\frac{\partial \ln [f(\lambda x)]}{\partial (\lambda)} = 1 \quad (11)$$

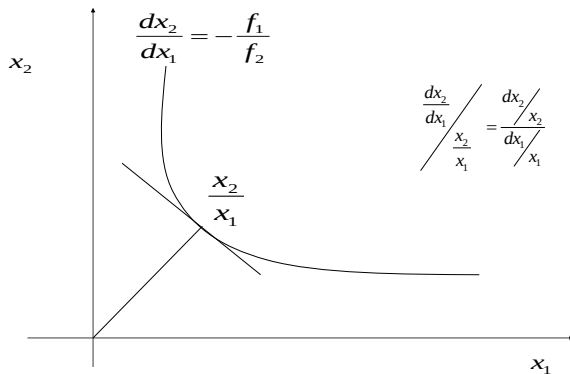
- If $\lambda = 1$ is to the right of A , then the production function exhibits decreasing returns to scale at x since any ray from the origin to $f(\lambda x)$ for $\lambda > A$ will cut $f(\lambda x)$ from below.
 - Thus, the *ray average product* is greater than the *ray marginal product*.
- If $\lambda = 1$ to the left of A , $f(x)$ exhibits increasing returns to scale at x .

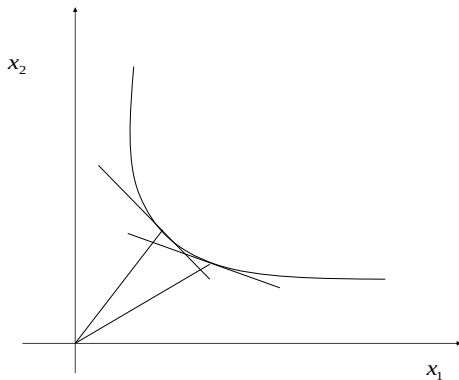
Measures of Input Substitution

- In the first lecture, we developed the idea of the *rate of technical substitution* defined as the movement along an isoquant.
- Now we want to expand our discussion to discuss an *elasticity of substitution*.
 - In general we would like to define the elasticity of substitution as the percentage change in relative rate of input use.
 - However, the exact nature of this elasticity is somewhat ambiguous.

- There are three general *elasticities of substitution*
 - Hicks defined the first elasticity of substitution in 1963. The Hicksian or direct elasticity of substitution

$$\sigma_{ij}^D = \frac{d\left(\frac{x_i}{x_j}\right) \frac{f_i}{f_j}}{d\left(\frac{f_i}{f_j}\right) \frac{x_i}{x_j}}. \quad (12)$$





Cobb-Douglas Example

- Using the Cobb-Douglas as an example

$$f(x) = Ax_1^\alpha x_2^\beta \Rightarrow \left\{ \begin{array}{l} f_1 = \alpha \frac{Ax_1^\alpha x_2^\beta}{x_1} \\ f_2 = \beta \frac{Ax_1^\alpha x_2^\beta}{x_2} \end{array} \right\} \Rightarrow \frac{f_1}{f_2} = \frac{\alpha x_2}{\beta x_1} \quad (13)$$

$$z(w) = \frac{f_1}{f_2}, \quad w = \frac{x_2}{x_1} \Rightarrow z(w) = \frac{\alpha}{\beta} w \Rightarrow \frac{dz(w)}{dw} = \frac{\alpha}{\beta} \quad (14)$$

$$\therefore \frac{d\left(\frac{x_2}{x_1}\right)}{d\left(\frac{f_1}{f_2}\right)} = \frac{\beta}{\alpha} \Rightarrow \sigma = \frac{\beta}{\alpha} \left(\frac{\frac{\alpha x_2}{\beta x_1}}{\frac{x_2}{x_1}} \right) = 1 \quad (15)$$

- Writing the bordered Hessian of the production surface

$$F = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{12} & f_{22} \end{vmatrix} \quad (16)$$

- This Hessian Represents the change in x_1 and x_2 such that y remains unchanged.

- Based on this transformation, the direct *elasticity of substitution* can be written as

$$\sigma_{12}^D = \frac{x_1 f_1 + x_2 f_2}{x_1 x_2} \frac{F_{12}}{F}$$

$$F_{12} = \begin{vmatrix} 0 & f_2 \\ f_1 & f_{12} \end{vmatrix} = -f_1 f_2$$

$$\begin{aligned} F &= \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{12} & f_{22} \end{vmatrix} = -f_1 \begin{vmatrix} f_1 & f_2 \\ f_{12} & f_{22} \end{vmatrix} + f_2 \begin{vmatrix} f_1 & f_2 \\ f_{11} & f_{12} \end{vmatrix} \\ &= -f_1 (f_1 f_{22} - f_2 f_{12}) + f_2 (f_1 f_2 - f_2 f_{11}) \\ &= -f_1^2 f_{22} + 2f_1 f_2 f_{12} - f_2^2 f_{11} \end{aligned} \quad (17)$$

Allen Partial Elasticity

- *Allen Partial Elasticity of Substitution* is a generalization of the matrix expression above

$$\sigma_{ij} = \frac{\sum_{i=1}^n x_i f_i}{x_i x_j} \frac{F_{ji}}{F}$$

$$F = \begin{vmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{12} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_1 & f_{1n} & f_{2n} & \cdots & f_{nn} \end{vmatrix} \quad (18)$$

Morishima Elasticity

- *Morishima Elasticity of Substitution* is the final generalization

$$\sigma_{ij}^M = \frac{f_j x_j}{f_i x_i} (\sigma_{ij} - \sigma_{jj}) \quad (19)$$