

Basic Notions of Production Functions

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Overview of Production Economics

The single chapter which follows summarizes the scope and method of agricultural production economics. The treatment does not do justice to this important field of applied economics, which is surpassed by no other in its accomplishment, its breadth, and its interesting history (Heady p.1).

Agricultural production economics is the oldest and most widespread specialization in the applied field of science known as agricultural economics; the science of agricultural economics originated as a study in farm production economics. The field has continued to grow until it now embraces more workers than any other specialization. In the field of agricultural economics the majority of outstanding names has been associated with some phase of production economics, both in farm and in national use of resources.

The list in the United States has included such prominent persons as G.F. Warren, H.C. Taylor, T.N. Carver, W.J. Spillman, L.C. Grey, C.L. Holmes, J.D. Black, E.C. Young, H. Tolley, M. Ezekiel, A. Boss, O.V. Wells, H.C.M. Case, S.E. Johnson, T.W. Schultz, G.E Forster, W.W. Wilcox, and W. G. Murray. The European list includes such well known names as J. Von Thunen, M. Kitchen, F Aereboe, T. Brinkman, A.W. Ashby, G. Lauer, R. Cohen and others. All of these scholars have been known for their findings or particular methods of analysis. In this list are not only persons who pioneered the field of agricultural economics but also those who contributed both to theory and to empirical knowledge. The central objective of their work has been to increase the efficiency with which farm resources are used (Heady p.1).

Production economists who focus their attention on agriculture are concerned with choice and decision-making in the use of the capital, labor, land, and management resources in the farming industry. The goals of agricultural production economics are twofold: (1) to provide guidance to individual farmers in using their resources most efficiently, and (2) to facilitate the most efficient use of resources from the standpoint of the consuming economy (Heady p.3).

Overview of Production Function

- The production function is a technical relationship depicting the technical transformation of inputs into outputs.
 - The production function in and of itself is devoid of economic content.
 - In the development of production functions, we are interested in certain characteristics that make it possible to construct economic models based on optimizing behavior.

Chambers on Production Functions

The production function (and indeed all representations of technology) is a purely technical relationship that is void of economic content. Since economists are usually interested in studying economic phenomena, the technical aspects of production are interesting to economists only insofar as they impinge upon the behavior of economic agents.... Because the economist has no inherent interest in the production function, if it is possible to portray and to predict economic behavior accurately without direct examination of the production function, so much the better. This principle, which sets the tone for much of the following discussion, underlies the intense interest that recent developments in duality have aroused (Chambers p.7).

The point of these two statements is that economists are not engineers and have no insights into why technologies take on any particular shape. We are only interested in those properties that make the production function that make the production function consistent with optimizing behavior.

Production Function Mapping

- One way to write the production function is as a function map

$$f : R_n^+ \rightarrow R_m^+ \quad (1)$$

which states that the production function (f) is a function that maps n inputs into m outputs.

- By convention, we are only interested in positive input bundles that yield positive output bundles.
- The first lecture will focus on the production function as a continuous function as students have probably seen it in previous courses. The next lecture will develop the concept of the production function more rigorously.

One Product, One Variable Factor

- A commonly used form of the production function is the "closed form" representation where *total physical product* is depicted as a function of a vector of inputs

$$y = f(x) \quad (2)$$

where y is the scalar (single) output and x is a vector of inputs.

- Focusing for a moment on the single output case, we could simplify the above representation to

$$y = f(x_1 | x_2) \quad (3)$$

or we are interested in examining the relationship between x_1 and y given that all other factors of production are held constant.

Derived Relationships

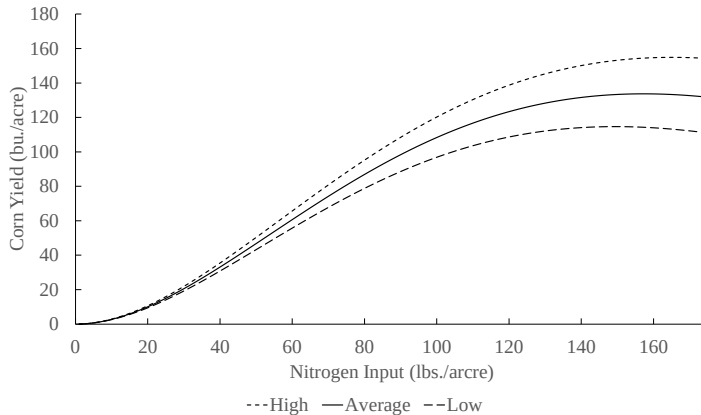
- Using this relationship, we want to identify three primary relationships:
 - Total Physical Product* – which is the original production function.
 - Average Physical Product* – defined as the average output per unit of input. Mathematically,

$$APP = \frac{y}{x} = \frac{f(x)}{x}. \quad (4)$$

- Marginal Physical Product* – defined as the rate of change in the *total physical product* at a specific level of input. Mathematically,

$$MPP = \frac{d(TPP)}{dx} = \frac{dy}{dx} = \frac{df(x)}{dx} = f'(x). \quad (5)$$

Three Production Functions

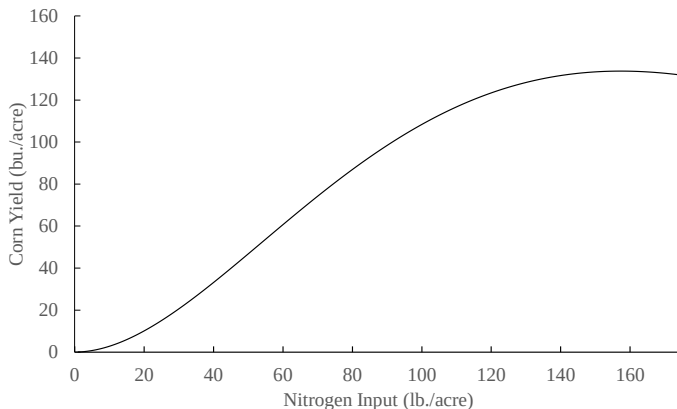


- This set of production functions are taken from Moss and Schmitz – Investing in Precision Agriculture.
- The shape is referred to as "sigmoid."
- The exact functional form can be attributed to Zellner, A. 1951. An Interesting General Form for a Production Function. *Econometrica* 19: 188-189.
- The exact mathematical form of the function is

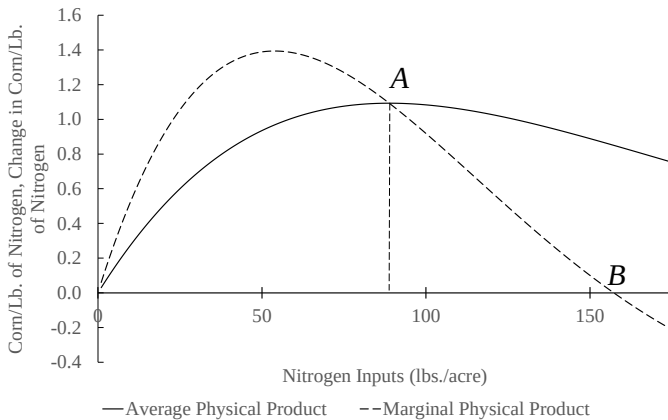
$$\phi(v_1, v_2) = \frac{av_1^3}{\exp\left[b\frac{v_1}{v_2}\right] - 1}. \quad (6)$$

- The average function sets $v_2 = 1.0$, $a = 0.0005433$, and $b = 0.01794$.

Production Function for Average Yield



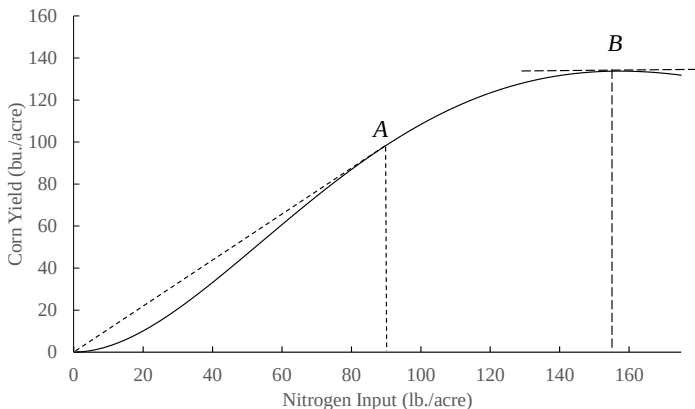
Average and Marginal Physical Product Graphs



Stages of Production

- *Stage I:* This stage of the production function is defined as that region where the average physical product is increasing. In this region, the marginal physical product is greater than the average physical product. In this region, each additional unit of input yields relatively more output on average.
- *Stage II:* This stage of the production process corresponds with the economically feasible region of production. Marginal physical product is positive and each additional unit of input produces less output on average.
- *Stage III:* This stage of production implies negative marginal return on inputs.

Stages of Production on Total Physical Product Graph

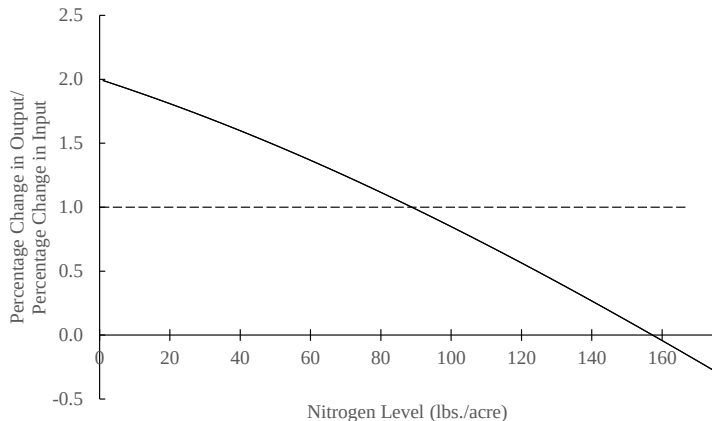


Elasticity of Production

- Elasticities are often used in economics to produce a unit-free indicator of the shape of a function. Most are familiar with the elasticity of consumer demand.
- In defining the production function, we are interested in the *factor elasticity*.
- The factor elasticity is defined as

$$E = \frac{\% \Delta y}{\% \Delta x} = \frac{\frac{dy}{y}}{\frac{dx}{x}} = \frac{dy}{dx} \frac{x}{y} = \frac{MPP}{APP} \quad (7)$$

Factor Elasticity for the Zellner Function



APP and MPP

- There is a specific relationship between the *average physical product* and *marginal physical product* when the average physical product is maximized. Mathematically,

$$MPP = \frac{dTPP}{dx} = \frac{d x APP}{dx} = APP + x \frac{d APP}{dx} \quad (8)$$

- Thus, when the APP is maximized

$$\frac{d APP}{dx} = 0 \Rightarrow MPP = APP. \quad (9)$$

- Following through this relationship, we have

$$\begin{aligned}\frac{d APP}{dx} &> 0 \Rightarrow MPP > APP \Rightarrow E > 1 \\ \frac{d APP}{dx} &= 0 \Rightarrow MPP = APP \Rightarrow E = 1 \\ \frac{d APP}{dx} &< 0 \Rightarrow MPP < APP \Rightarrow E < 1\end{aligned}\quad (10)$$

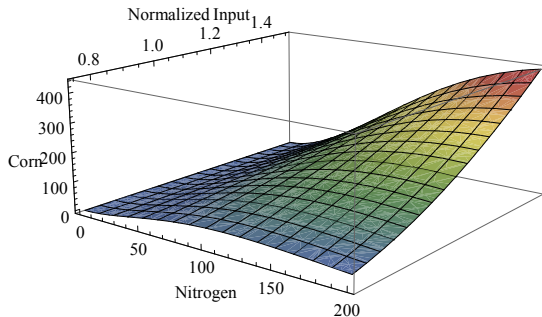
- In addition, we know that if $E = 0 \Leftrightarrow MPP = 0$, TPP is maximum and $E < 0 \Leftrightarrow MPP < 0$.
- Thus, if $E > 1$ then the production function is in Stage I. If $1 > E > 0$, then the production function is in Stage II. If $E < 0$, then the production function is in Stage III.

One Product, Two Variable Input Relationships

- Expanding the production, we start by considering the case of two inputs producing one output. In the general functional mapping notation

$$\begin{aligned} f : R_2^+ &\rightarrow R_1^+ \\ y &= f(x_1, x_2) \end{aligned} \tag{11}$$

Two Inputs



- These functions still have *average physical products* and *marginal physical products*, but they are conditioned on the level of other inputs.
- The *average physical products* can be written as

$$\begin{aligned} APP_1 &= \frac{y}{x_1} = \frac{f(x_1, x_2)}{x_1} \\ APP_2 &= \frac{y}{x_2} = \frac{f(x_1, x_2)}{x_2} \end{aligned} \tag{12}$$

- Similarly, the *marginal physical products* are defined by the partial derivatives

$$\begin{aligned} MPP_1 &= \frac{\partial y}{\partial x_1} = \frac{\partial f(x_1, x_2)}{\partial x_1} \\ MPP_2 &= \frac{\partial y}{\partial x_2} = \frac{\partial f(x_1, x_2)}{\partial x_2} \end{aligned} \tag{13}$$

Taylor Series Expansion

- It may be useful at this point to briefly visit the notion of the Taylor expansion.
- Taking the second-order expansion of the production function yields

$$f(x_1, x_2) \approx f(x_1^0, x_2^0) + \left[\frac{\partial f(x_1, x_2)}{\partial x_1} \quad \frac{\partial f(x_1, x_2)}{\partial x_2} \right] \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} & \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x_1, x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x_1, x_2)}{\partial^2 x_2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \quad (14)$$

- This approximation is exact in the case of either linear or quadratic production functions.
- However, if we focus on a quadratic production function, it is clear that

$$dy \approx f_1 \left\{ = \frac{\partial f(x_1, x_2)}{\partial x_1} \right\} dx_1 + f_2 \left\{ = \frac{\partial f(x_1, x_2)}{\partial x_2} \right\} dx_2 \quad (15)$$

Empirical Production Functions

- Linear Production Function

$$y = b_1x_1 + b_2x_2 \quad (16)$$

- Quadratic Production Function

$$y = a_1x_1 + a_2x_2 + \frac{1}{2} (A_{11}x_1^2 + 2A_{12}x_1x_2 + A_{22}x_2^2)$$
$$y = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (17)$$

- Cobb-Douglas Production Function

$$y = Ax_1^{b_1}x_2^{b_2} \quad (18)$$

- Transcendental Production Function

$$y = Ax_1^{a_1} e^{b_1 x_1} x_2^{a_2} e^{b_2 x_2} \quad (19)$$

- Constant Elasticity of Substitution (CES) Production Function

$$y = A \left[bx_1^{-g} + (1-b)x_2^{-g} \right]^{\frac{-v}{g}} \quad (20)$$

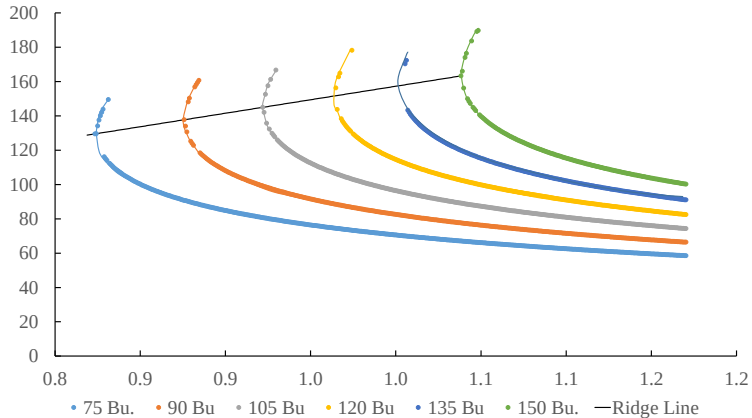
Isoquants, Isoclines, and Ridgelines

- Given the multivariate nature of the production function, it is possible to define isoquants, or the relationship that depicts the combinations of inputs that yield the same output.
- Starting from the basic production function

$$y = f(x_1, x_2) \Rightarrow x_2 = f^*(x_1, y) \quad (21)$$

- That is we are interested in constructing a functional mapping of x_2 based on the level of x_1 and y .
- Next, we hold the level of output constant and derive the levels of x_2 for any level of x_1 .

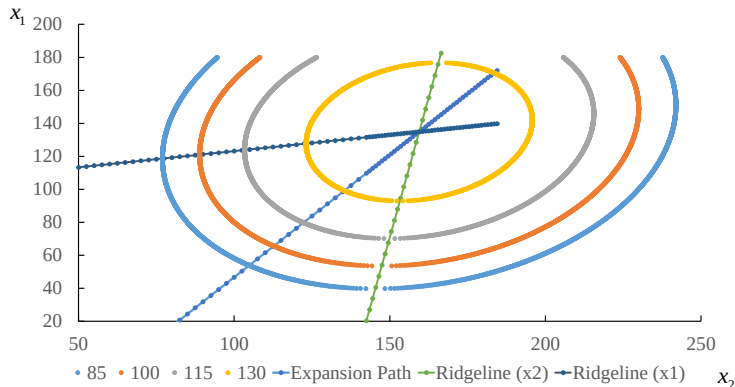
Isoquants



- The isoquants are useful in defining the rate of technical substitution or the rate at which one input must be traded for the other input

$$dy = f_1 dx_1 + f_2 dx_2 = 0 \Rightarrow \frac{dx_1}{dx_2} = -\frac{f_2}{f_1} \quad (22)$$

Ridgelines and Isoclines



Factor-Factor Relationships

- *Factor independence*: Two factors are independent if the MPP of one factor is not a function of the MPP of the other factor.
- The simplest example of this is a quadratic production function with $A_{12} = A_{21} = 0$. In this case, the isoquants are circles (or ellipses).

$$y = a_1x_1 + a_2x_2 + \frac{1}{2} (A_{11}x_1^2 + A_{22}x_2^2) \Rightarrow \begin{cases} \frac{\partial y}{\partial x_1} = a_1 + A_{11}x_1 \\ \frac{\partial y}{\partial x_2} = a_2 + A_{22}x_2 \end{cases} \quad (23)$$

- Case I: If $f_{12} > 0$, then x_1 and x_2 are technically complementary.

$$\frac{\partial^2 y}{\partial x_1 \partial x_2} = \frac{\partial \left(\frac{\partial y}{\partial x_1} \right)}{\partial x_2} = f_{12} > 0 \quad (24)$$

- Case II: If $f_{12} = 0$, then x_1 and x_2 are technically independent.
- Case III: if $f_{12} < 0$, then x_1 and x_2 are technically competitive.