

Basic Simplex Lecture IV

I. Some Problem Reformulation

A. The original problem:

$$\max z = 50x_1 + 10x_2$$

$$\text{s.t. } .3x_1 + .2x_2 \leq 25$$

$$x_1 + x_2 \leq 100$$

$$\max z - 50x_1 - 10x_2 = 0$$

$$\text{s.t. } .3x_1 + .2x_2 + s_1 = 25$$

$$x_1 + x_2 + s_2 = 100$$

In this formulation, we turn the inequalities constraints into equalities by adding slack variables. In order to maintain consistency, $s_1, s_2 \geq 0$. If we had a constraint such as

$$5x_1 + 4x_2 \geq 20$$

This equation would be transformed into an equality by

$$5x_1 + 4x_2 - s_1 = 20, s_1 \geq 0$$

In each equation we add a nonnegative slack (or surplus variable).

- B. Back to the original problem, we now have 2 constraint equations and 4 unknowns (see notes on Canonical form).
1. In general, you will have n unknowns and m equations with $m < n$.
 - a. $n - m$ variables will be nonzero in any basis or base solution (including the slack variables).
 2. A feasible basic solution is a basic solution satisfying the nonnegativity restrictions.
 3. Basic variables are those variables in a particular base solution that have nonzero values.
 4. Nonbasic variables are those variables at zero level in the current solution.
 5. Each basic solution is associated with an iteration.
 6. An entering variable at a particular iteration is the variable going from a value of zero to a nonzero value.
 7. An exiting variable is the variable going from a nonzero value to a zero value.

II. Steps in the Simplex Algorithm

- A. Using the standard form, compute an initial feasible or starting basic feasible solution.

$$\begin{aligned} \max z &= 50x_1 + 10x_2 \\ \text{s.t.} \quad &.3x_1 + .2x_2 \leq 25 \\ &x_1 + x_2 \leq 100 \end{aligned}$$

$$\begin{aligned} \max z &- 50x_1 - 10x_2 + 0s_1 + 0s_2 = 0 \\ \text{s.t.} \quad &.3x_1 + .2x_2 + s_1 = 25 \\ &x_1 + x_2 + s_2 = 100 \end{aligned}$$

Is this solution feasible? Do the equations hold with an equality and positive values for each variable?

	x_1	x_2	s_1	s_2	<i>RHS</i>
z	-50	-10	0	0	0
s_1	.3	.2	1	0	25
s_2	1	1	0	1	100

This is the L.P. simplex form a canonical form with $s_1 = 25$ and $s_2 = 100$.

- B. Select the leaving variable. The leaving variable is that variable with the most negative coefficient in the objective function. This implies the largest positive change in the objective function. In this case, x_1 has the largest negative coefficient.
- C. Select the leaving variable, that is select the basic variable that will become nonbasic. This is accomplished by finding the most constraining RHS.

	x_1	x_2	s_1	s_2	<i>RHS</i>
z	-50	-10	0	0	0
s_1	.3	.2	1	0	25
s_2	1	1	0	1	100

$25 / .3 = 250 / 3$
 $100 / 1 = 100$

s_1 , the slack variable for the labor constraint indicates that the most x_1 , corn production, that can enter the solution is 83.33 acres.

- D. Generate the next solution using Gauss-Jordon to find the change in basis.
1. This operation is called pivoting, the whole step is called an iteration. We are actually trying to find the next feasible solution in the direction of the optimum.
 2. The leaving row is called the pivot equation. The element at the intersection of the pivot row and the entering column is called the pivot element.
 3. Question: If we had a 3x3 system and the right-hand side how would we solve it?

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 2 & 0 & 1 & 2 \\ 0 & 3 & 2 & 1 \end{array} \right] R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & -2 & -3 & -4 \\ 0 & 3 & 2 & 1 \end{array} \right] R_2 \cdot \frac{1}{2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & \frac{3}{2} & 2 \\ 0 & 3 & 2 & 1 \end{array} \right] \begin{array}{l} R_1 - R_2 \\ R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & -\frac{5}{2} & -5 \end{array} \right] R_3 \cdot -\frac{2}{5}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 - \frac{3}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\begin{array}{l} x_1 + x_2 + 2x_3 = 3 \\ 0 + (-1) + 2(2) = 3 \end{array}$$

4. Divide the leaving row by the pivot element

	x_1	x_2	s_1	s_2	<i>RHS</i>
z					
s_1	1	$\frac{2}{3}$	$\frac{3}{10}$	0	83.333
s_2					

5. Compute the remaining tableau by:

Old equation or row element – Entering column coefficient* New pivot equation element

Example: Objective function row, cotton column

$$-50 - 50 \cdot 1 = 0$$

	x_1	x_2	s_1	s_2	<i>RHS</i>
z	$-50 - (-50 \cdot 1)$	$-10 - (-50 \cdot \frac{2}{3})$	$0 - (-50 \cdot \frac{3}{10})$	$0 - (-10 \cdot 0)$	$0 - (-50 \cdot \frac{250}{3})$
x_1	1	$\frac{2}{3}$	$\frac{3}{10}$	0	83.333
s_2	$1 - (1 \cdot 1)$	$1 - (1 \cdot \frac{2}{3})$	$0 - (1 \cdot \frac{3}{10})$	1	16.667

or

	x_1	x_2	s_1	s_2	<i>RHS</i>
z	0	23.333	15	0	4166.667
x_1	1	$\frac{2}{3}$	$\frac{3}{10}$	0	83.333
s_2	0	$\frac{1}{3}$	$-\frac{3}{10}$	1	16.667

6. Go back to step 2.
 - a. Optimality condition: If the nonbasic variables have positive coefficients in the maximization problem then you have reached an optimum.
 - b. Concepts in the tableau:
 - (i) Optimal solution: $x_1 = 83.333$, $s_2 = 16.667$.
 - (ii) Binding and nonbinding constraints.
 - (a) The first constraint is binding (it constraints the optimal).
 - (b) The second constraint is nonbinding (it does not constrain the optimum, adding an additional unit will not affect the objective function value).
 - (iii) Finally, we will consider the sensitivity of the solution, over what ranges will the current solution be optimal? What is the value of each unit of additional resource?

III. Next Example

$$\begin{aligned}
 \max z &= 40x_1 - 20x_2 + 0s_1 + 0s_2 + 0s_3 = 0 \\
 \text{s.t.} \quad &x_1 + x_2 + s_1 = 0 \\
 &6x_1 + 2x_2 + s_2 = 0 \\
 &36x_1 + 18x_2 + s_3 = 0
 \end{aligned}$$

A. In tableau form, this problem is expressed as

	x_1	x_2	s_1	s_2	s_3	RHS	
z	-40	-20	0	0	0	0	
s_1	1	1	1	0	0	12	$12/1 = 12$
s_2	6	2	0	1	0	48	$48/6 = 8$
s_3	36	18	0	0	1	360	$360/36 = 10$

1. Thus, s_2 is the leaving variable and x_1 is the entering variable

	x_1	x_2	s_1	s_2	s_3	RHS	
z	0	$-20/3$	0	$20/3$	0	320	
s_1	0	$2/3$	1	$-1/6$	0	4	$12/2 = 6$
x_1	1	$1/3$	0	$1/6$	0	8	$24/1 = 24$
s_2	0	6	0	-6	1	72	$72/6 = 7$

2. Thus, s_1 is the leaving variable and x_2 is the entering variable.

	x_1	x_2	s_1	s_2	s_3	RHS
z	0	0	10	5	0	360
x_2	0	1	$3/2$	$-1/4$	0	6
x_1	1	0	$-1/2$	$1/4$	0	6
s_3	0	0	3	$-9/2$	1	36