

Graphical Solution and Sensitivity (Again)

Lecture III

I. Problem from Last Time

$$\max z = 50x_1 + 10x_2$$

A. s.t. $.3x_1 + .2x_2 \leq 25$: Labor

$$x_1 + x_2 \leq 100$$
: Land

1. Labor

$$.3x_1 + .2x_2 = 25$$

$$.3x_1 = 25 - .2x_2$$

$$x_1 = \frac{250}{3} - \frac{2}{3}x_2$$

$$x_2 = 0 \Rightarrow x_1 = \frac{250}{3} \doteq 83.33$$

$$x_1 = 0 \Rightarrow x_2 = 125$$

2. Land

$$x_1 + x_2 = 100$$

$$x_1 = 100 - x_2$$

$$x_2 = 0 \Rightarrow x_1 = 100$$

$$x_1 = 0 \Rightarrow x_2 = 100$$

3. Objective

$$z = 50x_1 + 10x_2$$

$$50x_1 = z - 10x_2$$

$$x_1 = \frac{z}{50} - \frac{1}{5}x_2$$

4. How does the objective function change as x_1 and x_2 change?

$$z = Ax_1 + Bx_2$$

$$dz = Adx_1 + Bdx_2$$

B. Where do the labor and land constraints intersect?

$$\frac{250}{3} - \frac{2}{3}x_2 = 100 - x_2$$

$$\frac{1}{3}x_2 = \frac{50}{3}$$

$$x_2 = 50$$

$$x_2 = 50 \Rightarrow x_1 = 100 - 50 = 50$$

- C. What is the maximum allowable profit per acre of cotton such that the current solution remains optimal? Similarly, what is the minimum return on cotton for which this solution remains optimal?

1. Reformulate the objective function

$$z = z_1 x_1 + z_2 x_2$$

$$z_1 x_1 = z - z_2 x_2$$

$$x_1 = \frac{z}{z_1} - \frac{z_2}{z_1} x_2$$

Assuming z_2 is fixed at 10, what level of z_1 would cause the optimum allocation to change?

2. Any increase in the value of z will result in no change in the optimal solution.
3. The range over which the solution is stable in the other direction is bounded by the labor constraint. Specifically, if the objective function becomes more steep than the labor constraint, then the optimal solution will change. To solve for the coefficient at which the solution changes

$$\frac{10}{z_1} x_1 = \frac{2}{3} x_1$$

$$\frac{30}{2} = z_1$$

$$15 = z_1$$

Thus, the coefficient on cotton could decline to 15, a decrease of \$35/acre before the optimal solution changes.

4. Similarly for the coefficient on corn, the coefficient on corn could be decreased indefinitely without changing the optimal solution. However, if the coefficient were increased sufficiently corn would enter the solution

$$\frac{z_2}{50} x_2 = \frac{2}{3} x_2$$

$$3z_2 = 100$$

$$z_2 \doteq 33.33$$

Thus, if the rate of return per acre of corn were increased to \$33.33/acre, then corn would enter the optimal solution.

II. New Problem

- A. The firm's problem is how to organize production {choose levels of crop plantings} to maximize profits or net returns over variable costs given 12 acres of land, 48 hours of labor, and \$360 of operating capital.

Constraints	Corn x_1	Oats x_2
Land	1	1
Labor	6	2
Capital	36	18
Returns/Acre	\$40	\$20

B. Specification of the mathematical programming model:

$$\max_{x_1, x_2} 40x_1 + 20x_2$$

$$s.t. \quad x_1 + x_2 \leq 12: \text{Land}$$

$$6x_1 + 2x_2 \leq 48: \text{Labor}$$

$$36x_1 + 18x_2 \leq 360: \text{Capital}$$

1. Land

$$x_1 + x_2 \leq 12$$

$$x_1 = 12 - x_2$$

$$x_1 = 12 \Rightarrow x_2 = 0$$

$$x_1 = 0 \Rightarrow x_2 = 12$$

2. Labor

$$6x_1 + 2x_2 \leq 48$$

$$6x_1 = 48 - 2x_2$$

$$x_1 = 8 - \frac{1}{3}x_2$$

$$x_1 = 8 \Rightarrow x_2 = 0$$

$$x_1 = 0 \Rightarrow x_2 = 24$$

3. Capital

$$36x_1 + 18x_2 \leq 360$$

$$36x_1 = 360 - 18x_2$$

$$x_1 = 10 - \frac{1}{2}x_2$$

$$x_1 = 10 \Rightarrow x_2 = 0$$

$$x_1 = 0 \Rightarrow x_2 = 20$$

4. Where do these various constraints intersect?

a. Land n Labor

$$12 - x_2 = 8 - \frac{1}{3}x_2$$

$$4 = \frac{2}{3}x_2$$

$$x_2 = 6, x_1 = 6$$

b. Capital n Labor

$$10 - \frac{1}{2}x_2 = 8 - \frac{1}{3}x_2$$

$$2 = \frac{1}{6}x_2$$

$$x_2 = 12, x_1 = 4$$

c. Capital n Land

$$12 - x_2 = 10 - \frac{1}{2}x_2$$

$$2 = \frac{1}{2}x_2$$

$$x_2 = 4, x_1 = 8$$

C. Optimal combination of crops

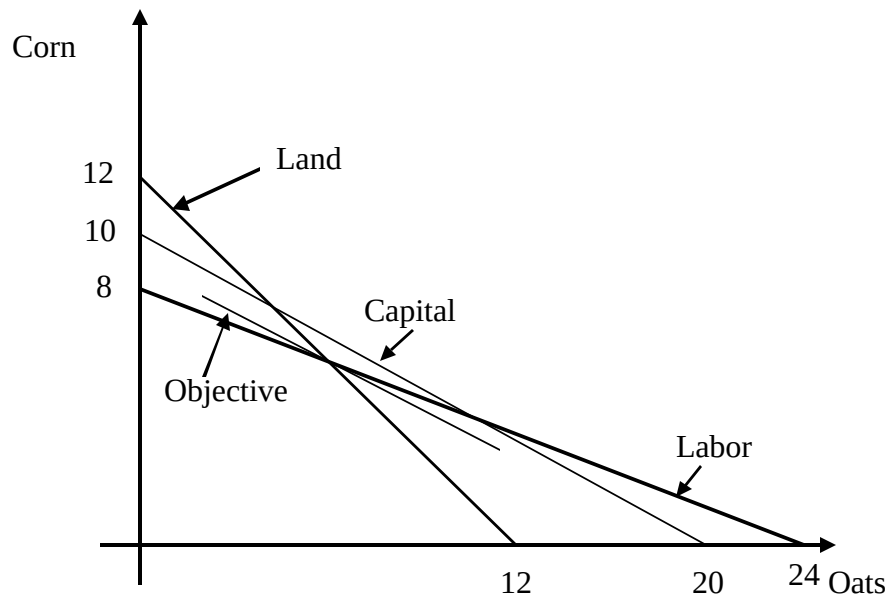
$$z = 40x_1 + 20x_2$$

$$40x_1 = z - 20x_2$$

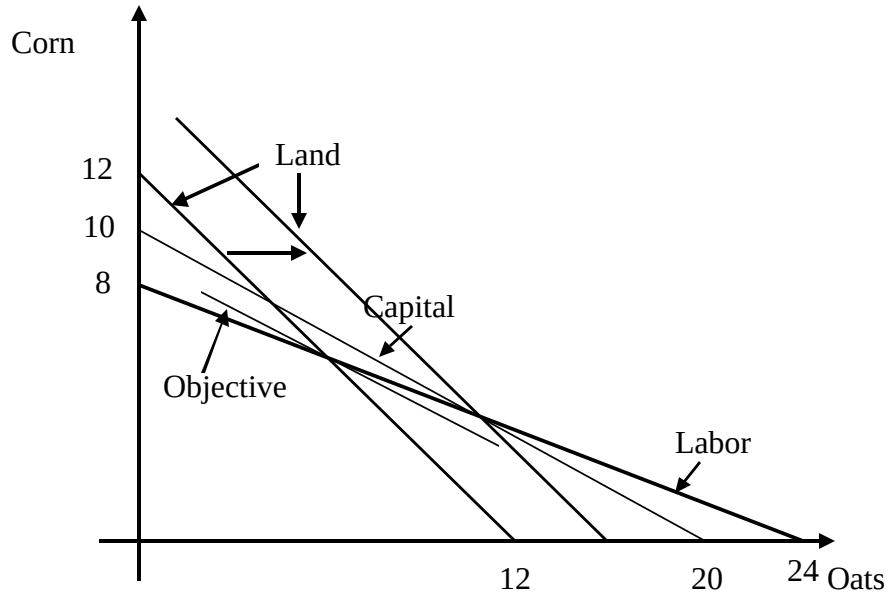
$$x_1 = \frac{z}{40} - \frac{1}{2}x_2$$

This implies an optimal solution at the corner of the Land n Labor constraints:

$$-1 < -\frac{1}{2} < -\frac{1}{3}$$



- D. Sensitivity Analysis—Over what ranges are the solution stable?
 1. What if land were increased? What constraint would become binding?



If land were sufficiently increased, then the combination of labor and capital will eventually become constraining. The constrained optimal at that point involves $x_1=4$ and $x_2=12$ or 16 acres of land (an increase of 4 acres). The value of the objective function at this point is:

$$z = 40(4) + 20(12) = 160 + 240 = 400$$

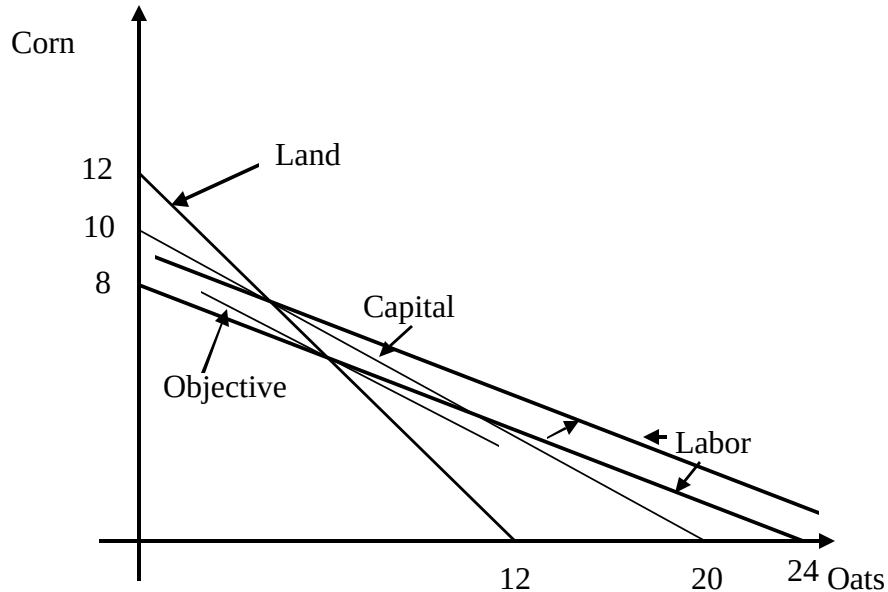
$$ShadowValue = \frac{400 - 360}{16 - 12} = \frac{40}{4} = 10$$

On the down side, land can be decreased until the labor constraint is no longer binding before the solution changes. At that point $x_1=8$ and $x_2=0$ (a decrease of 8 acres of land)

$$z = 40(8) + 20(0) = 320$$

$$ShadowValue = \frac{320 - 360}{8 - 12} = \frac{-40}{-4} = 10$$

2. What if the labor constraint were increased? What constraint would become binding?



Increasing the labor would cause the Capital n Land intersection to become binding. Thus, the largest change in labor that would keep the constraint active is $x_1=8$ and $x_2=4$

$$NewLabor: 6(8) + 2(4) = 48 + 8 = 56$$

$$z = 40(8) + 20(4) = 320 + 80 = 400$$

$$ShadowValue = \frac{400 - 360}{56 - 48} = \frac{40}{8} = 5$$

The labor constraint can likewise be reduced until the point where the land is restricted without change in the active constraints.

3. Capital could be decreased until the land and labor intersection is binding at $x_1=6$ and $x_2=6$.
- E. Over what range of objective coefficients is the solution stable?

$$z = z_1x_1 + z_2x_2$$

$$x_1 = \frac{z}{z_1} - \frac{z_2}{z_1}x_2$$

$$Fixing z_2 = 20$$

$$-\frac{20}{z_1} = -1 \Rightarrow z_1 = 20, \Delta z_1 = 20 - 40 = -20$$

$$-\frac{20}{z_1} = -\frac{1}{3} \Rightarrow z_1 = 60, \Delta z_1 = 60 - 40 = 20$$

Fixing $z_1 = 40$

$$\frac{z_2}{40} = 1 \Rightarrow z_2 = 40, \Delta z_2 = 40 - 20 = 20$$

$$\frac{z_2}{40} = \frac{1}{3} \Rightarrow z_2 = \frac{40}{3}, \Delta z_2 = 13.33 - 20 = -6.67$$