

The Basic Math Programming Model and Graphical Approach

Lecture II

I. An Agricultural Example

- A. Assume that the farmer can produce two outputs, cotton and corn, with 100 acres of land and 25 hours of labor. Further, assume that each crop has the following input use per acre:

	Cotton	Corn
Labor	.3	.2
Land	1.0	1.0

Finally, assume that the profit per acre is \$50 for cotton and \$10 for corn.

- B. Basic Construction of the Problem: The basic formulation of the problem involves answering three questions.

1. What does the model seek to determine—What are the variables?
2. What constrains the variables from taking on any value—What are the constraints?
3. What is the objective? – What is the rule used to determine the best or optimum?

- C. In the current model determine each component.

- D. Basic Mathematical Formulation

$$\max 50x_1 + 10x_2$$

$$s.t. \quad .3x_1 + .2x_2 \leq 25$$

$$x_1 + x_2 \leq 100$$

$$x_1, x_2 \geq 0$$

where x_1 is cotton and x_2 is corn.

- E. Graphical solution

1. Basic Formulation of constraints

$$.3x_1 + .2x_2 = 25$$

$$.3x_1 = 25 - .2x_2$$

$$x_1 = \frac{250}{3} - \frac{2}{3}x_2$$

$$x_1 + x_2 = 100$$

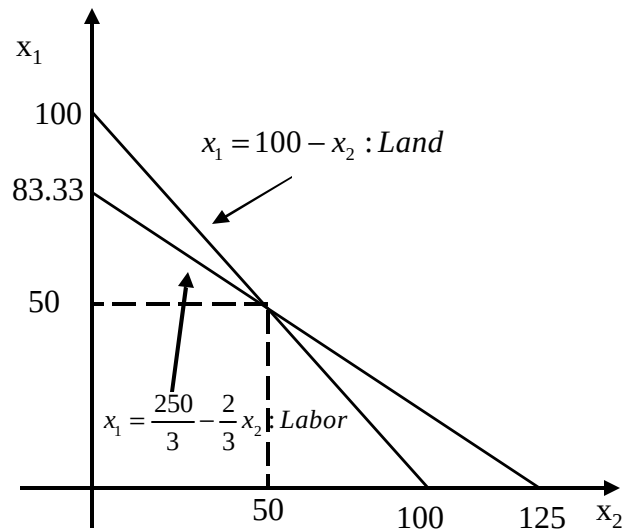
$$x_1 = 100 - x_2$$

Do these lines intersect?

$$\frac{250}{3} - \frac{2}{3}x_2 = 100 - x_2$$

$$\frac{1}{3}x_2 = \frac{50}{3}$$

$$x_2 = 50$$



The region satisfying all constraints is called the feasible region or region of feasibility.

2. Choosing the optimal point: Construction of the Iso-profit line. Let z be the profit from some allocation of cotton and corn. Specifically,

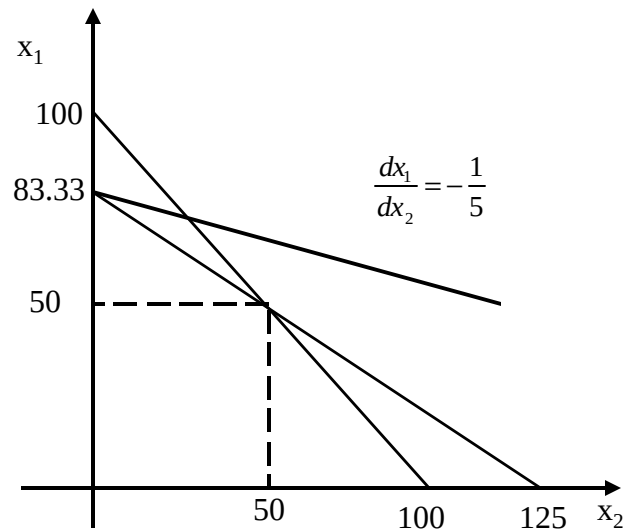
$$z = 50x_1 + 10x_2$$

Question: How do we know that the objective value increases as x_1 and x_2 increase? Using basic calculus, we differentiate the profit function and set it equal to zero yielding

$$dz = 50dx_1 + 10dx_2 = 0$$

$$\Rightarrow \frac{dx_1}{dx_2} = -\frac{1}{5}$$

Thus, we know that profit increases as either x_1 or x_2 increases by the first equation. Second, we know what the effect of increasing x_1 on profit is given that we have to decrease x_2 . For example, on the land constraint (where we have to trade one unit of x_1 for each unit of x_2) the effect on profit is $-1/5$. Alternatively, profit is maximized where the slope of the feasible region is equal to $-1/5$.



3. Two things to note:
 - a. First, the trick is to find the rightmost iso-objective function line that is still in feasible or solution space.
 - b. Second, notice that the solution is on a corner or extreme point.

II. A Slight Reformulation: Slack Variables and Matrix Algebra

A. The complicating factor in the above formulation from simple linear algebra perspective is the inequalities.

1. Note that these inequalities can be transformed into equalities if we add nonnegative slack variables:

$$\max 50x_1 + 10x_2$$

$$\text{s.t. } .3x_1 + .2x_2 + s_1 = 25$$

$$x_1 + x_2 + s_2 = 125$$

$$x_1, x_2, s_1, s_2 \geq 0$$

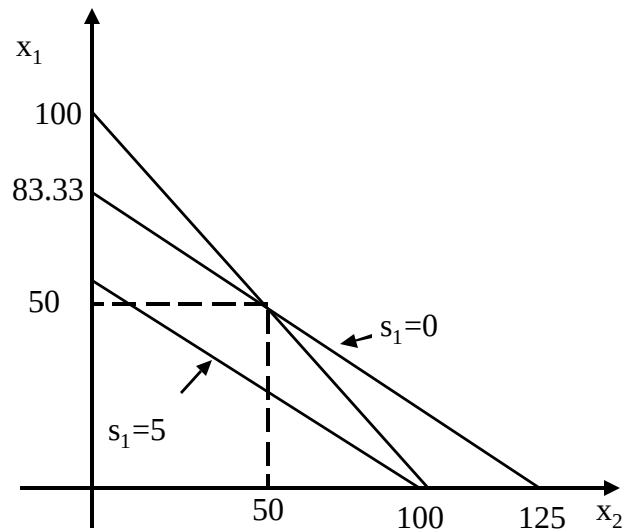
- a. s_1 is the slack or excess variable in the labor constraint while s_2 is the slack or excess variable in the land constraint.

- b. Graphical solution

$$.3x_1 + .2x_2 + s_1 = 25$$

$$.3x_1 = 25 - .2x_2 - s_1$$

$$x_1 = \frac{250}{3} - \frac{2}{3}x_2 - \frac{10}{3}s_1$$



- c. Why $s_1=5$? For a second, what if $x_1=0$ and $x_2=100$. Then

$$\begin{aligned} \frac{250}{3} - \frac{200}{3} \left\{ \frac{2}{3} x_2 \right\} - \frac{10}{3} s_1 &= 0 \\ \frac{50}{3} - \frac{10}{3} s_1 &= 0 \\ \frac{10}{3} s_1 &= \frac{50}{3} \\ s_1 &= 5 \end{aligned}$$

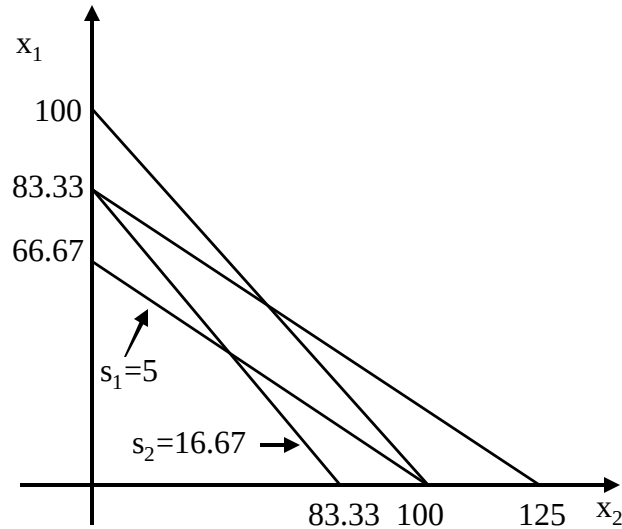
Or, more appropriately, suppose $s_1=5$. Then

$$\begin{aligned} x_1 &= \frac{250}{3} - \frac{2}{3} x_2 - \frac{10}{3} (5) \\ x_1 &= \frac{200}{3} - \frac{2}{3} x_2 \end{aligned}$$

- d. Two things to note:
- (i) We can transform an inequality system into an equality system by adding (subtracting) the appropriate slack (surplus) variables.
 - (ii) The nonnegativity condition on the slack (surplus) variable limits the feasibility condition to everything below (above) the line
- e. The land constraint

$$\begin{aligned} x_1 + x_2 + s_2 &= 100 \\ x_1 &= 100 - x_2 - s_2 \end{aligned}$$

2. Combining the two equality representations graphically



- Thus, one alternative is to produce $x_1=83.33$ acres and $x_2=0$ acres. This exhausts the labor constraint ($s_1=0$), but leaves land in excess ($s_2=16.67$).
- Another alternative is to use all the land and let labor go slack: $x_2=100$, $x_1=0$, $s_1=5$, and $s_2=0$.

B. Using Matrix Algebra

- Note that an important part of the solution process was determining the intersection of the two constraints. This process can be simplified using matrix algebra.
 - Note that the system of equations

$$.3x_1 + .2x_2 = 25$$

$$x_1 + x_2 = 100$$

can be expressed in matrix form as

$$\begin{bmatrix} .3 & .2 & | & 25 \\ 1 & 1 & | & 100 \end{bmatrix}$$

using row operations, we can multiply the first row by $10/3$ yielding

$$\begin{bmatrix} 1 & 2/3 & | & 250/3 \\ 1 & 1 & | & 100 \end{bmatrix}$$

Subtracting the first row from the second,

$$\begin{bmatrix} 1 & 2/3 & | & 250/3 \\ 0 & 1/3 & | & 50/3 \end{bmatrix}$$

Multiplying the last row by 3,

$$\left[\begin{array}{cc|c} 1 & 2/3 & 250/3 \\ 0 & 1 & 50 \end{array} \right]$$

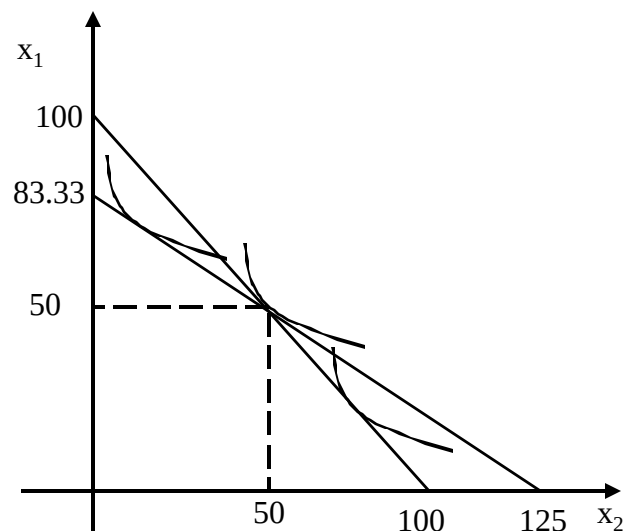
Subtracting $2/3$ times the second row from the first yields

$$\left[\begin{array}{cc|c} 1 & 0 & 50 \\ 0 & 1 & 50 \end{array} \right]$$

which is interpreted as $x_1=50$, $x_2=50$.

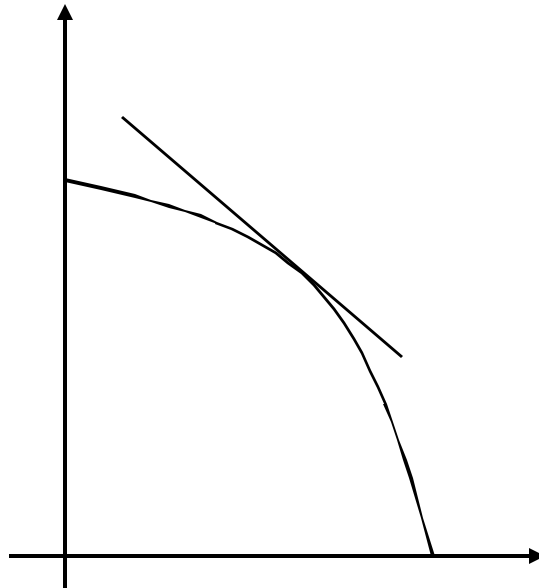
III. Stuff from Day

- A. The preceding problem was a simple example of linear programming. It contained linear constraints, a single linear objective function, and continuous variables. Many extensions of this framework are now possible.
1. The simplest extension is the nonlinear objective function.

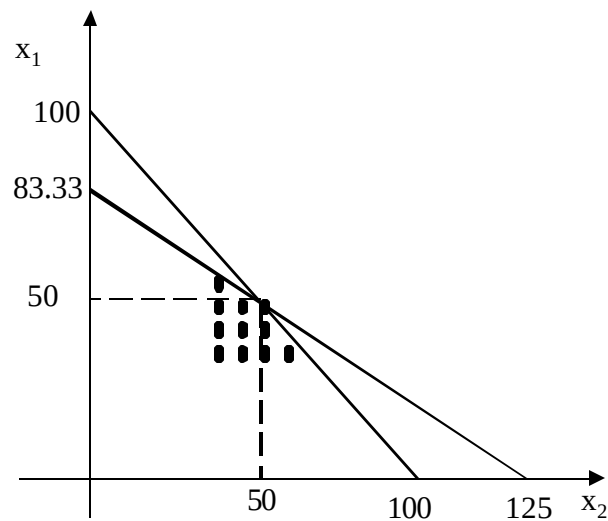


The nonlinear objective function allows for an optimum not on a corner. It may also produce an optimum without bounds.

2. Next, nonlinear constraints have much the same effect as a nonlinear objective function

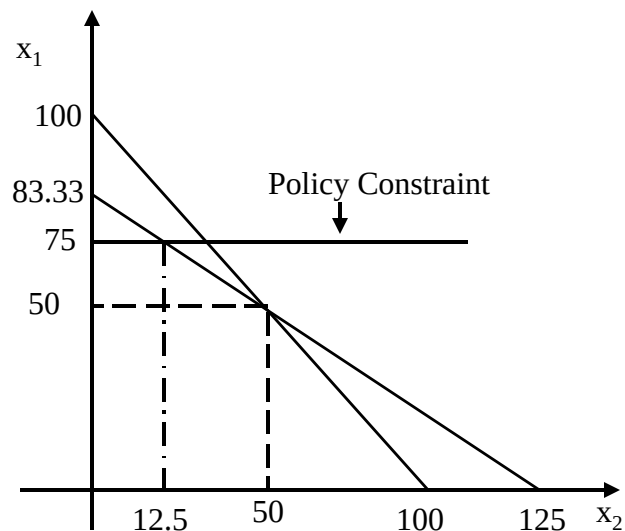


3. Integer Programming: Integer programming is the branch of mathematical programming which searches for whole number solutions. This type of programming became popular when it became difficult to raise 1.236 steer calves. The feasible region is developed in much the same way as the basic problem, but only integer solutions are considered.



IV. Sensitivity

- A. In and of itself, the solution of an optimum may be of little value. More important questions may involve the tendency of the solution to change as certain parameters of the model change. Three types of these changes are
1. Changes in resource availability over which the same constraints are binding.
 2. What is the value of an additional unit of resource?
 3. By how much can a given objective function coefficient change before the solution changes?
- B. Adjusted agricultural problem:
1. Additional constraints: Assume that the current government program restricts the producer to growing no more than 75 acres of cotton, and no less than 25 acres of cotton.
 - a. Does the policy change the solution? Yes, given that the current solution for the producer to grow 83.33 acres of cotton, we know that the producer will reduce cotton by 8.33 acres.
 - b. Graphically, the new feasible region can be depicted as:



- c. The government policy changes the solution in that in the original solution the cotton was constrained by the labor constraint. By constraining the cotton acreage, labor is freed. The question then becomes, can this freed labor be used to increase profit? The answer is yes.
 - (i) What is the quantity of labor freed?

$$.3(83.33 - 75) = s_1$$

$$.3(8.33) = 2.5 = s_1$$

- (ii) How much corn could be grown with that freed labor?

$$x_2 = 2.5 \left(\frac{10}{2} \right) = 12.5$$

- (iii) Would this violate another constraint?

$$x_1 \{= 75.0\} + x_2 \{= 12.5\} = 87.5 \leq 100$$

2. What increase in labor would be required to violate the current set of constraints?
- a. Given the government program, labor could be added until land becomes binding, or $x_1=75$ and $x_2=25$.

$$.3(75) + .2(25) = 27.5$$

- b. How much could land be reduced before the current set of binding constraints are changed? 77.5 Acres.
3. What is the cost of each change?

$$z^*[75, 12.5] = 75(50) + 12.5(10) = 3,775$$

$$z^*[83.33, 0] = 4,167 = z^*[75, 12.5] + 392$$

$$z[75, 25] = 4,000 = z[83.33, 0] - 225$$

Thus, profit would be increased by \$392 if the government constraint on cotton were lifted, and by \$225 if more labor were available.

4. What is the value of a change in resource?

$$\text{Labor} : \frac{225}{27.5 - 25} = 90$$

$$\text{Allotment} : \frac{392}{83.33 - 75} = 47.04$$