

Lecture X: Implications and Empirical Tests of the Capital Asset Pricing Model

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November 18, 2013

I. Properties of the CAPM

- A. In equilibrium, all asset returns will fall on the security market line (SML) whether the asset is on the capital market line (CML) or not.
 - 1. In our derivation we solved w_t , σ_t , σ_t^2 and $\bar{z}_t$ which represented some point on the ES frontier as depicted in Figure 1.
 - 2. In all probability this market portfolio is a combination of several assets.
 - 3. The SML is a derived relationship between returns on individual assets and their betas, or the ratio of the covariance to the variance on the market portfolio as depicted in Figure 2.
 - 4. Bivariate derivation of the CAPM

$$\begin{aligned} E[R_p] &= aE[R_i] + (1-a)E[R_m] \\ \sigma(R_p) &= [a^2\sigma_i^2 + (1-a)^2\sigma_m^2 + 2a(1-a)\sigma_{im}]^{1/2} \end{aligned} \quad (1)$$

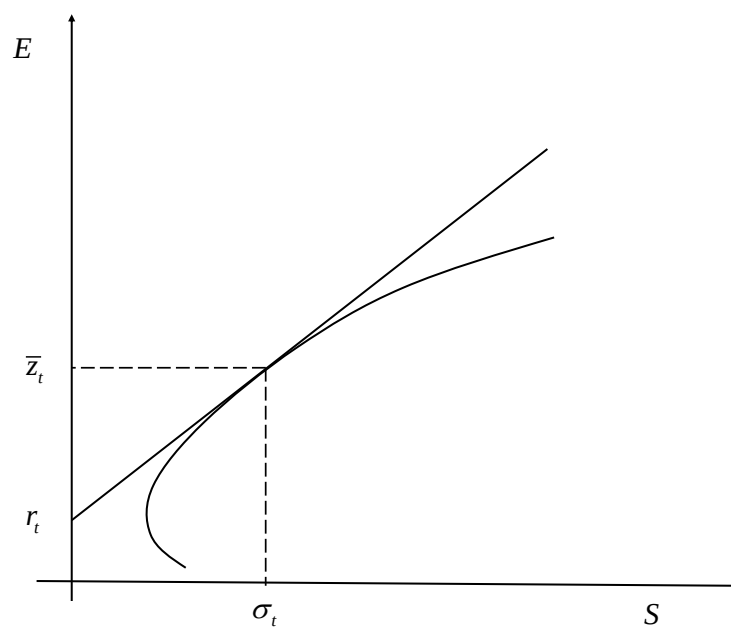


Figure 1: E-S Frontier

$$\begin{aligned}
 \frac{\partial E[R_p]}{\partial a} &= E[R_i] - E[R_m] \\
 \frac{\partial \sigma(R_p)}{\partial a} &= \frac{1}{2} \left[a^2 \sigma_i^2 + (1-a)^2 \sigma_m^2 + 2a(1-a) \sigma_{im} \right]^{-1/2} \cdot \\
 &\quad [2a\sigma_i - 2\sigma_m^2 + 2a\sigma_m^2 + 2\sigma_{im} - 4a\sigma_{im}] \\
 &= \frac{1}{\sigma_p} [a\sigma_i^2 - \sigma_m^2 + a\sigma_m^2 + \sigma_{im} - 2a\sigma_{im}] \\
 \left. \frac{\partial \sigma(R_p)}{\partial a} \right|_{a=0} &= \frac{\sigma_{im} - \sigma_m^2}{\sigma_m} \\
 &= \left[\frac{\sigma_{im} - \sigma_m^2}{\sigma_m} \right] \frac{\sigma_m}{\sigma_m} = [\beta - 1] \sigma_m
 \end{aligned} \tag{2}$$

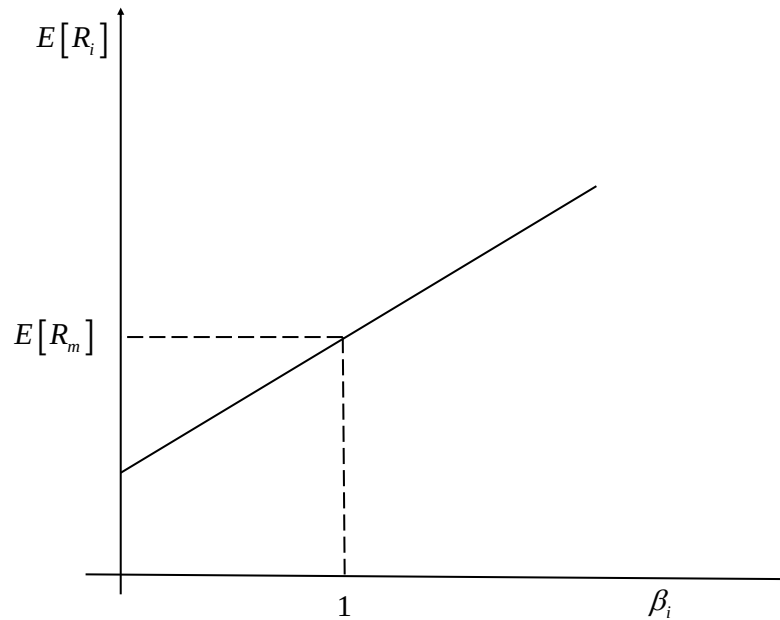


Figure 2: Security Market Line

$$\begin{aligned} \therefore \frac{\partial E[R_p] / \partial a}{\partial \sigma(R_p) / \partial a} \bigg|_{a=0} &= \frac{E[R_i] - E[R_m]}{[\beta - 1] \sigma_m} \\ &= \frac{1}{[\beta - 1]} \frac{E[R_i] - E[R_m]}{\sigma_m} \end{aligned} \quad (3)$$

The slope of the CML is

$$\begin{aligned} &\frac{E[R_m] - r_f}{\sigma_m} \\ \therefore \left[\frac{1}{\beta - 1} \right] \left[\frac{E[R_i] - E[R_m]}{\sigma_m} \right] &= \frac{E[R_m] - r_f}{\sigma_m} \end{aligned} \quad (4)$$

$$\begin{aligned} E[R_i] - E[R_m] &= (\beta - 1) [E[R_m] - r_f] \\ E[R_i] - E[R_m] &= \beta [E[R_m] - r_f] - E[R_m] + r_f \\ E[R_i] - r_f &= \beta [E[R_m] - r_f] \end{aligned}$$

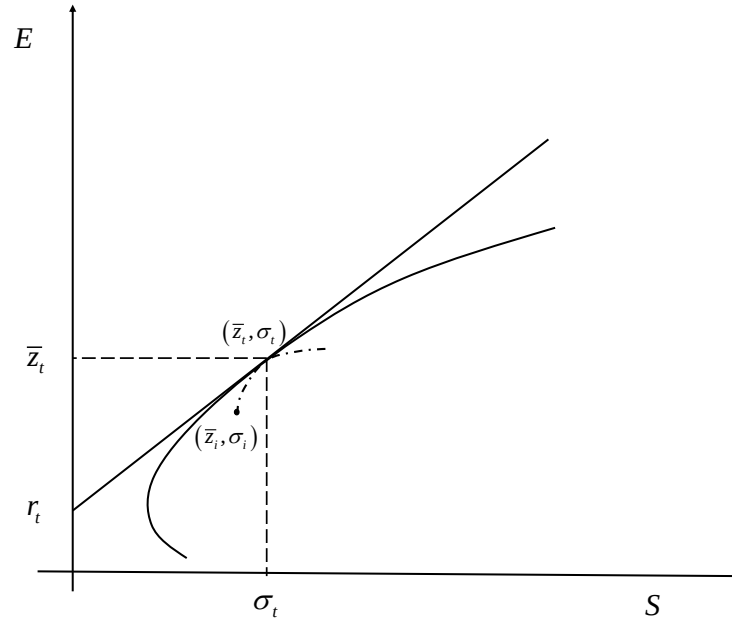


Figure 3: Mean and Return for an Individual Asset

- a. The advantage to this proof, if you will, is the ability to depict the SML tradeoff on the CML space.
- b. Suppose we wanted to examine asset i that was off the efficient frontier as depicted in Figure 3.
 - i. A portfolio consisting of only asset i will lie on the interior of the ES frontier. The combination of this single asset portfolio and the efficient portfolio is a quadratic function between the two points.
 - ii. In general, the slope of that quadratic is

$$\frac{\partial E[R_p] / \partial a}{\partial \sigma(R_p) / \partial a} = \frac{E[R_i] - E[R_m]}{\frac{(a\sigma_i^2 - \sigma_m^2 + a\sigma_m^2 + \sigma_{im} - 2a\sigma_{im})}{(a^2\sigma_i^2 + (1-a)^2\sigma_m^2 + 2a(1-a)\sigma_{im})}} \quad (5)$$

- iii. AS the portfolio of the efficient portfolio and the single asset portfolio approaches the efficient portfolio, the slope of this frontier becomes

$$\lim_{a \rightarrow 0} \frac{\partial E[R_p] / \partial a}{\partial \sigma(R_p) / \partial a} = \frac{1}{\beta - 1} \frac{E[R_i] - E[R_m]}{\sigma_m} \quad (6)$$

In other words, as the portfolio approaches the ES frontier, the ES tradeoff becomes a function of the asset's β .

- iv. The classical CAPM expression is then derived by equating this slope with the CML slope.
5. It is hoped that the forgoing discussion will lend support to the decomposition of return risk into diversifiable or nonsystematic and nondiversifiable or systematic risk.
 - a. The empirical model used with the CAPM is

$$\begin{aligned} \tilde{R}_j &= a_j + b_j \tilde{R}_m + \tilde{\epsilon}_j \\ \sigma_j^2 &= E[R_j^2] - \bar{R}_j^2 \\ &= E[a_j + b_j \tilde{R}_j + \tilde{\epsilon}_j]^2 - [a_j + b_j \bar{R}_m]^2 \\ &= E[a_j^2 + 2a_j b_j \tilde{R}_m + 2a_j \tilde{\epsilon}_j + b_j^2 \tilde{R}_m^2 + 2\tilde{\epsilon}_j b \tilde{R}_m + \tilde{\epsilon}_j^2] \\ &\quad - [a_j^2 + b_j^2 \bar{R}_m^2 + 2a_j b_j \bar{R}_m] \\ &= a_j^2 + 2a_j b_j \bar{R}_m + b_j^2 E[\tilde{R}_m^2] + E[\tilde{\epsilon}_j^2] - a_j^2 - b_j^2 \bar{R}_m^2 - 2a_j b_j \bar{R}_m \\ &= b_j^2 [E[\tilde{R}_m^2] - \bar{R}_m^2] + E[\tilde{\epsilon}_j^2] \\ \sigma_j^2 &= b_j^2 \sigma_m^2 + \sigma_\epsilon^2 \end{aligned} \quad (7)$$

- b. This equality states that the total riskiness of the asset can be decomposed into systematic or nondiversifiable risk ($\beta_j^2 \sigma_m^2$) and nonsystematic or diversifiable risk.
- B. A second important property of the CAPM is that the measure of risk for individual assets is linearly additive

$$\beta_p = a\beta_x + b\beta_y \quad (8)$$

1. Remember $\beta_i = \sigma_{im} / \sigma_m^2$

$$\begin{aligned}
 \therefore \beta_p &= \frac{E[(ax + by - aE[x] - bE[y])(R_m - E[R_m])]}{\sigma_m^2} \\
 &= \frac{E[(a(x - E[x]) + b(y - E[y]))(R_m - E[R_m])]}{\sigma_m^2} \\
 &= \frac{E[a(x - E[x])(R_m - E[R_m]) + b(y - E[y])(R_m - E[R_m])]}{\sigma_m^2} \\
 &= \frac{aE[(x - E[x])(R_m - E[R_m])] + bE[(y - E[y])(R_m - E[R_m])]}{\sigma_m^2} \\
 &= a\beta_x + b\beta_y
 \end{aligned} \tag{9}$$

II. Uses of the CAPM valuations

A. Single-Period Models

1. The rate of return on a single holding period is denoted

$$\tilde{R}_j = \frac{\tilde{P}_e - P_0}{P_0} \tag{10}$$

where P_e is the end of period price and P_0 is the beginning period price.

2. From the CAPM model

$$\begin{aligned}
 E[R_j] &= R_f + [E[R_m] - R_f] \beta = \frac{E[\tilde{P}_e] - P_0}{P_0} \\
 R_f + [E[R_m] - R_f] \beta &= \frac{E[\tilde{P}_e]}{P_0} - 1 \\
 1 + R_f + [E[R_m] - R_f] \beta &= \frac{E[\tilde{P}_e]}{P_0}
 \end{aligned} \tag{11}$$

$$P_0 = \frac{E[\tilde{P}_e]}{1 + R_f + [E[R_m] - R_f] \beta}$$

This formulation is the **Risk Adjusted Discount Rate**.

3. An alternative approach is called the certainty equivalent approach

$$\begin{aligned}
 1 + R_f + [E[R_m] - R_f] \beta &= \frac{E[\tilde{P}_e]}{P_0} \\
 1 + R_f &= \frac{E[\tilde{P}_e] - [E[R_m] - R_f] \beta P_0}{P_0} \\
 P_0 &= \frac{E[\tilde{P}_e] - [E[R_m] - R_f] \beta P_0}{1 + R_f}
 \end{aligned} \tag{12}$$

Copeland and Weston replace the last term with $\lambda \text{Cov}(\tilde{P}_e, \tilde{R}_m)$ where $\lambda = [E[R_m] - \bar{R}_m] / \sigma_m^2$. The argument is

$$\begin{aligned}
 \text{Cov} [\tilde{R}_j, \tilde{R}_m] &= \text{Cov} \left[\frac{\tilde{P}_e - P_0}{P_0}, \tilde{R}_m \right] \\
 &= \text{E} \left[\left(\frac{\tilde{P}_e - P_0}{P_0} - \frac{\text{E} [\tilde{P}_e] - P_0}{P_0} \right) (\tilde{R}_m - \text{E} [\tilde{R}_m]) \right] \quad (13) \\
 &= \frac{1}{P_0} \text{E} \left[(\tilde{P}_e - \text{E} [\tilde{P}_e]) (\tilde{R}_m - \text{E} [\tilde{R}_m]) \right] \\
 &= \frac{1}{P_0} \text{Cov} (\tilde{P}_e, \tilde{R}_m)
 \end{aligned}$$

- B. An alternative application of CAPM estimates to decision making involves investment ranking using the SML.
1. Assume that the corporation has fit the SML using its own stock values and possibly the stock values for other companies as depicted in Figure 4
 2. Next, suppose that the firm has an investment opportunity that has an (E, β) combination K . Should the company adopt this asset? –No– Why?

III. Empirical Tests of the CAPM

- A. The typical estimation procedure for empirically testing the CAPM is a two step model.
1. First, the annual returns are estimated as a function of the returns on the market portfolio

$$R_{jt} = a_j + b_j R_{mt}. \quad (14)$$

This estimation gives the estimate of the individual beta parameters. We would expect $b_j > 0$ for all stocks indicating a positive price of risk.

2. Using the estimated results from the first equation, the SML is estimated using cross sectional equations

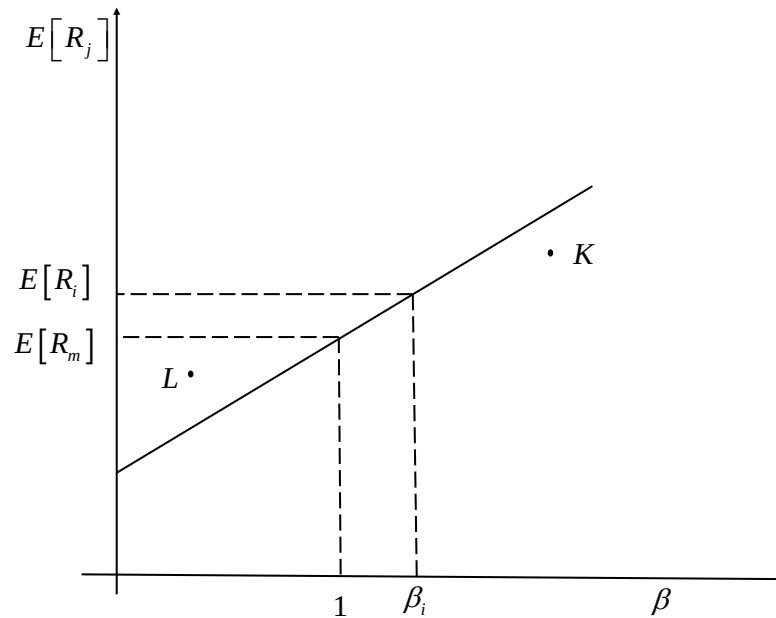


Figure 4: Different Returns to Corporation

$$\bar{R}_j = \hat{\gamma}_0 + \hat{\gamma}_1 b_j + u_j. \quad (15)$$

This regression estimates whether the betas are consistent across stocks.

B. The "testable" implications of the CAPM are:

1. The intercept term $\hat{\gamma}_0$ should be equal to zero.
2. Beta should be the only factor that explains the rate of return on a risky asset. Other terms such as residual variance, dividend yield, price/earnings ratio, firm size, or beta squared should be equal to zero.
3. The relationship in beta should be linear.
4. The coefficient $\hat{\gamma}_1$ should be equal to $R_{mt} - R_f$.
5. When the equation is estimated over long periods of time the rate of return on the market portfolio should be greater than the risk-free rate of return.

C. Some Empirical Results

1. Lintner, J. 1965. Security Prices and Risk: The Theory of Comparative Analysis of AT & T and Leading Industrials. Paper presented at the Conference on the Economics of Regulated Utilities, Chicago, Illinois, June.
 - a. Lintner examined the rate of returns for 301 stocks over the period 1954-63. He first estimated the b_i s and the $\bar{R}_i$ s.
 - b. Next, he fit the model

$$\bar{R}_i = \hat{\gamma}_0 + \hat{\gamma}_1 b_i + \hat{\gamma}_2 s_{ei}^2 + u_i \quad (16)$$

where b_i is the first pass estimate of the beta and s_{ei}^2 is an estimate of the residual variance based on the first pass residuals. If the CAPM model held we would expect $\hat{\gamma}_2 = 0$.

- c. The estimation results yielded the following results.
 - i. There was a positive and significant relationship between $\bar{R}_j$ and b_j . Unfortunately, the slope was lower than expected. This result coupled with the result that $\hat{\gamma}_0$ was higher than expected implies that the SML was rotated as depicted in Figure 5.
 - ii. The $\hat{\gamma}_2$ coefficient on the residual squared was positive and significant which contradicts CAPM.

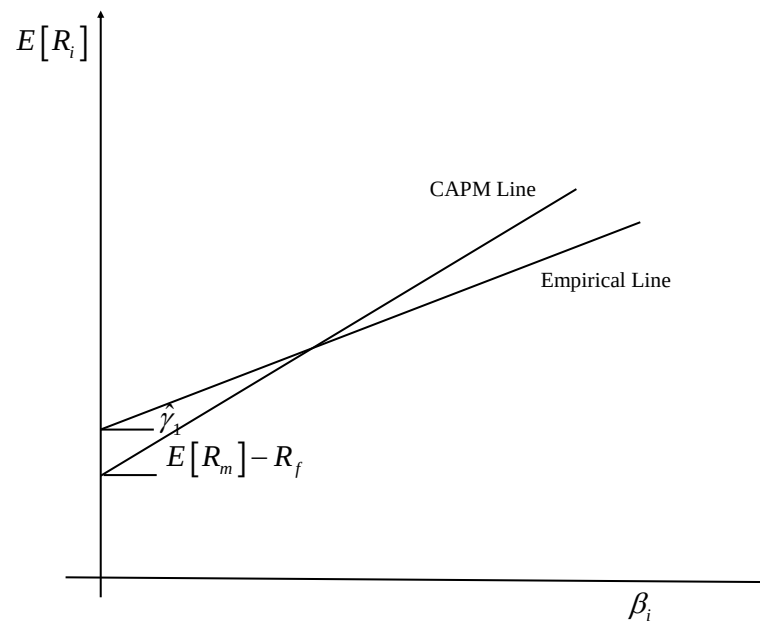


Figure 5: Lintner's Empirical Results