

Lecture VII: State-Preference Theory

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I. Overview

- A. Firms borrow capital for investment in real assets by selling securities; individuals obtain claims to the firm's assets by investing in securities. Thus, securities present opportunities for intertemporal shifts in consumption through time by financing productive activities.
 - 1. Individual consumption/investment decisions that determine aggregate security demand and firm investment decisions are both affected by security prices.
 - 2. By equating security supply and demand, security prices yield a consistent set of firm and investment decisions.
- B. The choice of securities is determined by the individual's desire to obtain some consumption pattern over a period of time.
 - 1. Typically, this period involves uncertainty. Thus, when an investor chooses an investment, he has actually chosen some distribution of future returns. Specifically, the value of the security or amount of consumption is dependent on a state of the economy.
 - 2. The uncertain value of a security can then be expressed as a vector of probable payoffs at some future date. The individual's portfolio then becomes a matrix of possible payoffs on different securities that make up the portfolio

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1N} \\ y_{21} & y_{22} & \cdots & y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ y_{S1} & y_{S2} & \cdots & y_{SN} \end{bmatrix} \quad (1)$$

- a. States are indexed $s = 1, \dots, S$.
- b. Assets are indexed $n = 1, \dots, N$.
3. In the state-preference model, uncertainty is the inability to determine which state you will be in at some future date. Thus, a security represents a claim to a vector or a bundle of state-contingent payoffs.
4. The set of states must be mutually exclusive and exhaustive.

II. Definition of Pure Securities

- A. Conceptually, the typical security has some positive payoff in any state of nature. However, the payoff vector can be viewed as a linear combination of primitive investments. A primitive investment is an investment that yields \$ 1 in state s and 0 in all other states. These primitive investments are called Arrow-Debreu securities.
- B. The existence of Arrow-Debreu securities is linked to the concept of a complete capital market. If a capital market is complete, it is possible to value every state

$$\begin{array}{ll} \text{risk free} & \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \\ \text{unemployment insurance} & \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \\ \text{risky debt} & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{array} \quad (2)$$

Note that risk-free less unemployment insurance is equal to risky debt, or

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0 \quad (3)$$

Hence A^{-1} $\nexists$.

- C. To complete the market, suppose you had a set of Arrow-Debreu securities

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4)$$

If you wanted to create a security with payoff (a, b, c) you could buy a units of $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, b units of $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, and c units of $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

- D. Thus, given complete capital markets we could theoretically reduce uncertainty about future wealth to zero.

III. Derivation of Pure Securities

- A. A useful approach is to use a parallel example. Suppose you have a company that sells fruit baskets with bananas and apples presented in Table 1. What is the price of bananas and apples? Notice that the total price times the pieces of fruit in the first basket must sum to \$ 8. Thus,

Table 1: Market for Fruit

Basket	Bananas	Apples	Price
Basket 1	10	20	\$ 8
Basket 2	30	10	\$ 9

$$10p_B + 20p_A = 8$$

$$30p_B + 10p_A = 9$$

$$\Rightarrow \begin{bmatrix} 10 & 20 \\ 30 & 10 \end{bmatrix} \begin{bmatrix} p_B \\ p_A \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \quad (5)$$

$$\Rightarrow \begin{bmatrix} p_B \\ p_A \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 30 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0.20 \\ 0.30 \end{bmatrix}$$

- B. Thus, in the capital market assume that security j pays \$ 10 under state 1 and \$ 20 under state 2. Further assume that security k

pays \$ 30 in state 1 and \$ 10 in state 2 (presented in Table 2). Thus, if p_s is the price of the Arrow-Debreu security in state s and Q_{js} represents the quantity of the pure security held in state s

Table 2: Security Market

Security	State 1	State 2	Price
j	\$ 10	\$ 20	\$ 8
k	\$ 30	\$ 10	\$ 9

$$\begin{aligned} p_1 Q_{j1} + p_2 Q_{j2} &= P_j \\ p_1 Q_{k1} + p_2 Q_{k2} &= P_k \end{aligned}$$

$$\begin{bmatrix} Q_{j1} & Q_{j2} \\ Q_{k1} & Q_{k2} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} P_j \\ P_k \end{bmatrix}$$

(6)

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} Q_{j1} & Q_{j2} \\ Q_{k1} & Q_{k2} \end{bmatrix}^{-1} \begin{bmatrix} P_j \\ P_k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 30 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0.20 \\ 0.30 \end{bmatrix}$$

IV. No Arbitrage Condition

- A. In capital market equilibrium the market prices are set so that supply equals demand. In the state-preference framework, one condition necessary for market equilibrium requires that any two portfolios with the same state-contingent payoff vectors must be priced identically. This condition is often called the single price law of markets.
- B. If short selling is allowed in the capital market we can obtain a second related necessary condition from the market equilibrium – the absence of any riskless arbitrage opportunity. For example, suppose that two portfolios with identical payoffs in all states sold for different prices $p_A > p_B$. Then it would be possible to sell

short B and buy A . The result would be a riskless profit in the current period $p_A - p_B$.

- C. In a perfect and complete capital market, any market security payoff vector can be exactly replicated by a portfolio of pure securities. Thus, it follows that when short selling is allowed, the no-arbitrage profit condition requires that the price of a security be equal to the price of the linear combinations of the pure securities that replicates the market security's payoff vector.

V. Economic Determinants of Security Prices

- A. The prices of pure securities will be shown to depend on:
1. Time preferences for consumption and the productivity of capital.
 2. Expectations as to the probability that a particular state will occur.
 3. Individual's attitudes toward risk given the variability across states of aggregate end-of-period wealth.
- B. The linkage between time preferences and the price of a pure security can be most easily seen by constructing a riskless asset.
1. A riskless asset is an asset that yields the same return in each state. For example, assume that the asset yields one dollar in each of three states.
 2. The price of this riskless asset is then $p_1 + p_2 + p_3 = 0.8$ or the sum of the prices of the pure securities.
 3. The price of the riskless claim to a dollar at the end of the period is just the present value of a dollar the risk-free rate r

$$\frac{1}{1+r} = p_1 + p_2 + p_3 = 0.8 = \frac{1}{1+0.25} \quad (7)$$

In general, $1/(1+r) = \sum_s p_s$.

4. The actual size of the interest rate will reflect individual time preferences for consumption and the productivity of capital.
- C. The second determinant of security prices is the individual's beliefs regarding the relative likelihood of each state π_s . This feature causes differences in security prices.

1. Beliefs are called state probabilities, π_s .
2. In principle, the subjective beliefs regarding the likelihood of each state can vary across individuals. However, we typically assume that individuals agree on the relative likelihoods or possess homogeneous expectations.
3. Given homogeneous expectations, the price of the pure securities can be decomposed into the probability of the state (π_s) and the price of an expected dollar in state s (θ_s) – $P_s = \pi_s \times \theta_s$.
 - a. The price of a pure security is equal to the expected end-of-period payoff discounted at its expected rate of return

$$P_s = \frac{1}{(1 + E[R_s])} \pi_s \quad (8)$$

where $0 < P_s < 1$

$$1 + E[R_s] = \frac{1}{P_s} \pi_s$$

$$E[R_s] = \frac{1}{\frac{P_s}{\pi_s}} - 1 \quad (9)$$

where $P_s/\pi_s = \theta_s$

$$E[R_s] = \frac{1}{\theta_s} - 1 \quad \ni: 0 < \theta_s < 1. \quad (10)$$

- b. So if the θ_s s were identical across states, the expected rates of return would be equal for all securities. But given that the probabilities across states differ, the expected payoffs across securities must differ.
- D. The third determinant of security prices and another reason for differences in those prices are individual's attitudes toward risk.
 1. If individuals are risk averse they will diversify their holdings in an attempt to reduce the variability of consumption in the next period.
 2. In fact, if the price of a dollar contingent on state s (θ_s) were equal in each state, then risk averse investors would invest

equal amounts of capital in each pure security. This would eliminate uncertainty about future wealth.

3. However, not everyone can do this since aggregate wealth is not the same in each state – there is nondiversifiable risk in the economy.
 - a. For example, suppose that the future economy possessed three possible states. The first state was a less prosperous outcome which yielded \$ 1 billion, the second state was average yielding \$ 2 billion, and the third was a boom economy which yielded \$ 3 billion. On average, the investors in the economy have to hold portfolios of $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \times x/6$ where x is the average wealth of each agent. Thus, even if the individuals are risk averse, the wealth in each state must be different. To assume these differences, θ_s changes between states. Specifically, θ_s must be high in the first state when wealth is relatively more scarce.
 - b. If aggregate wealth is the same in some states then risk averse investors would want to hold the same number of pure securities for those states and there would be no reason for prices of expected dollar payoffs to be different in those states. Investors would not want to hold unequal numbers of claims to the states with the same aggregate wealth because this would mean bearing risk that could be diversified away and there is no reason to expect reward for bearing diversifiable risk.

E. In summary security prices can be decomposed into three factors

$$\begin{aligned}
 P_s &= \pi_s \theta_s = \frac{\$ \pi_s}{1 + E[R_s]} \\
 &= \frac{\$1}{1 + r} \pi_s \left[\frac{1 + r}{1 + E[R_s]} \right] \\
 &= \frac{1}{1 + r} \pi_s \left[1 - \frac{E[R_s] - r}{1 + E[R_s]} \right]
 \end{aligned} \tag{11}$$

- a. $1/(1 + r)$ is the discount rate on a riskless asset.
- b. π_s is the probability of each state.

c. $1 - (E[R_s] - r) / (1 + E[R_s])$ is a risk adjustment factor.

VI. Optimal Portfolio Decisions

A. Given the state-preference framework it is possible to formulate the portfolio choice problem in expected utility fashion.

1. Because any security can be expressed as a combination of Arrow-Debreu securities assume that the portfolio choice problem is simply a problem of choosing the Arrow-Debreu securities that maximize utility

$$\begin{aligned} \max_{Q_s, C} & U(C) + \sum_s \pi_s U(Q_s) \\ \text{st} & \sum_s P_s Q_s + \$1C = W_0 \end{aligned} \quad (12)$$

2. Forming the Lagrangian

$$\begin{aligned} L &= U(C) + \sum_s \pi_s U(Q_s) + \lambda \left(W_0 - \sum_s P_s Q_s - \$C \right) \\ L_{Q_t} &= \pi_t U'(Q_t) - \lambda P_t = 0 \\ \Rightarrow \lambda &= \frac{\pi_t U'(Q_t)}{P_t} \\ L_C &= U'(C) - \$1\lambda = 0 \\ \Rightarrow \lambda &= \frac{U'(C)}{\$1} \\ L_\lambda &= W_0 - \sum_s P_s Q_s - \$1C = 0 \\ \Rightarrow \lambda &= \frac{\pi_t U'(Q_t)}{P_t} = \frac{U'(C)}{\$1} \end{aligned} \quad (13)$$