

Lecture VI: The Security Market Line, Equivalence between Mean-Variance and Expected Utility, and Introduction to State-Space Models

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I. Security Market Line

A. From Ingersoll, we know that

$$\bar{z}_t - R1 = \frac{\mathcal{Q}_t^2}{\sigma_t^2 [\bar{z}_t - R]} \quad (1)$$

What does this mean and how is it derived?

1. First by introducing a risk free asset to the investment portfolio, the set of risky assets was reduced to a single efficient risky asset. The portfolio created by leveraging or lending based on this risky asset dominates the old frontier.
2. The single risky asset is

$$w_t = \frac{\Sigma^{-1}(\bar{z} - R1)}{B - AR} \quad (2)$$

The covariance between the single efficient risky asset and the single assets in the portfolio then becomes

$$\begin{aligned}
 \varrho_t &= \Sigma w_t = \Sigma \left[\frac{\Sigma^{-1} (\bar{z} - R1)}{B - AR} \right] = \frac{(\bar{z} - R1)}{[B - AR]} \\
 \sigma_t^2 &= w_t' \varrho_t = w_t' \left[\frac{\bar{z} - R1}{B - AR} \right] \\
 &= \frac{w_t' \bar{z} - R w_t' 1}{B - AR} \\
 \Rightarrow \sigma_t^2 &= \frac{\bar{z}_t - R}{B - AR} \\
 B - AR &= \frac{\bar{z}_t - R}{\sigma_t^2} \\
 \therefore \varrho_t^2 &= \frac{\bar{z}_t - R1}{\left[\frac{\bar{z}_t - R}{\sigma_t^2} \right]} \\
 [\bar{z}_t - R] \frac{\varrho_t^2}{\sigma_t^2} &= \bar{z}_t - R1
 \end{aligned} \tag{3}$$

Typically, we denote $\beta \equiv \varrho_t^2 / \sigma_t^2$.

3. Thus, the optimal portfolio that causes the ray from the risk free rate of return to be tangent to the E-S frontier is such that the excess expected rate of return on each asset is proportional to the covariance of the assets returns with the optimal portfolio.

B. Graphically from Levy and Sarnat 1

II. Equivalence between Mean Variance and the Expected Utility Hypothesis

- A. Start by assuming that the decision maker has a negative exponential utility function

$$E[U(x)] = \int_{-\infty}^{\infty} -\exp[-\rho x] \frac{1}{\sigma \sqrt{2\pi}} \exp\left[-\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2}\right] dx \tag{4}$$

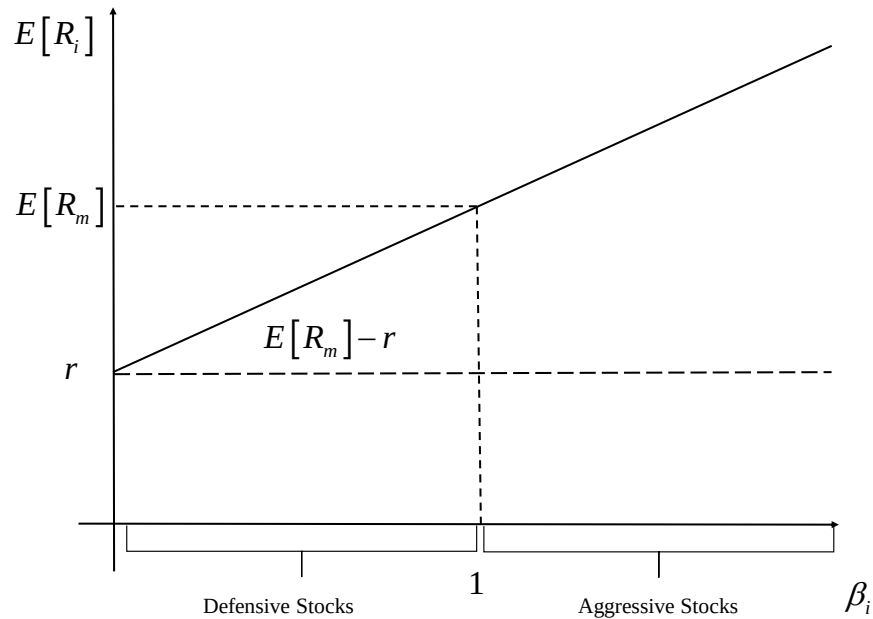


Figure 1: Graphic Analysis of Stock Beta

Change in variables – Let $z = (x - \mu) / \sigma$ then

$$\begin{aligned}\sigma z &= x - \mu \\ x &= \sigma z + \mu \\ dx &= \sigma dz\end{aligned}\tag{5}$$

The integration then becomes

$$\begin{aligned}\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} -\exp[-\rho(\sigma z + \mu)] \exp\left[-\frac{1}{2} \frac{(\sigma z + \mu - \mu)^2}{\sigma^2}\right] \sigma dz \\ = -\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\rho z \sigma - \rho \mu - \frac{1}{2} z^2\right] \sigma dz\end{aligned}\tag{6}$$

Notice that $-\rho z \sigma - \frac{1}{2} z^2$ is "close" to a square. Specifically,

$$\begin{aligned} & \frac{1}{2}\rho^2\sigma^2 + \rho z\sigma + \frac{1}{2}z^2 \\ & \frac{1}{2}(\rho^2\sigma^2 + 2\rho\sigma z + z^2) \end{aligned} \quad (7)$$

$$\frac{1}{2}(\rho\sigma + z)^2$$

Hence, we complete the square by adding $+\rho^2\sigma^2 - \rho^2\sigma^2$

$$\begin{aligned} & -\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[-\rho\mu + \frac{\rho^2\sigma^2}{2} - \frac{\rho^2\sigma^2}{2} - \rho z\sigma - \frac{z^2}{2} \right] \sigma dz \\ & -\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[\rho\mu + \frac{\rho^2\sigma^2}{2} - \frac{1}{2}(\rho\sigma + z)^2 \right] \sigma dz \quad (8) \\ & -\frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\rho\mu + \frac{\rho^2\sigma^2}{2} \right] \int_{-\infty}^{\infty} \exp \left[-\frac{1}{2}(\rho\sigma + z)^2 \right] \sigma dz \end{aligned}$$

Transforming the integral back

$$\begin{aligned} & z = \frac{x - \mu}{\sigma}, \quad dz = \frac{1}{\sigma} dx \\ \Rightarrow & -\frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\rho \left(\mu - \frac{\rho}{2}\sigma^2 \right) \right] \int_{-\infty}^{\infty} \exp \left[-\frac{1}{2} \left[\rho\sigma + \frac{x - \mu}{\sigma} \right]^2 \right] dx \quad (9) \\ = & -\frac{1}{\sigma\sqrt{2\pi}} \exp \left[-\rho \left(\mu - \frac{\rho}{2}\sigma^2 \right) \right] \int_{-\infty}^{\infty} \exp \left[-\frac{1}{2} \left[\frac{x + \rho\sigma^2 - \mu}{\sigma} \right]^2 \right] dx \end{aligned}$$

Defining $x^* = x + \rho\sigma^2$ (shifting the mean of the distribution)

$$\begin{aligned}
 & -\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\rho\left(\mu - \frac{\rho}{2}\right)\right] \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left[\frac{x^* - \mu}{\sigma}\right]^2\right] dx^* \\
 & \ni: \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left[\frac{x^* - \mu}{\sigma}\right]^2\right] dx = 1 \quad (10) \\
 & \therefore E[U(x)] = -\exp\left[-\rho\left(\mu - \frac{\rho}{2}\sigma^2\right)\right]
 \end{aligned}$$

Thus, if preferences are negative exponential and the distribution is normal, expected utility can be exactly represented by the mean-variance frontier.