

# Lecture Vb: Mean Variance – Introduction of Risk-Free Assets, and Equivalence between Expected Utility and Mean-Variance

Charles B. Moss

April 11, 2013

## I. Introduction of a Risk-Free Asset

A. If a risk-free asset is introduced into the portfolio, the efficient set of portfolios becomes a straight line between the risk-free asset and a tangency on the Expected Value-Standard Deviation frontier.

B. Mechanics for Lecture 05

$$\min \frac{1}{2} w' \Sigma w \tag{1}$$

$$\text{st } (\bar{z} - R1)' w = \mu - R$$

with  $w_0^* = 1 - 1'w$  as the amount investment in the risk-free asset if  $w_0^* > 0$  or borrowed if  $w_0^* < 0$  the optimal portfolio becomes

$$\gamma = \frac{\mu - R}{C - 2RB + R^2A} \tag{2}$$

$$w^* = \frac{\mu - R}{C - 2RB + R^2A} \Sigma^{-1} (\bar{z} - R1)$$

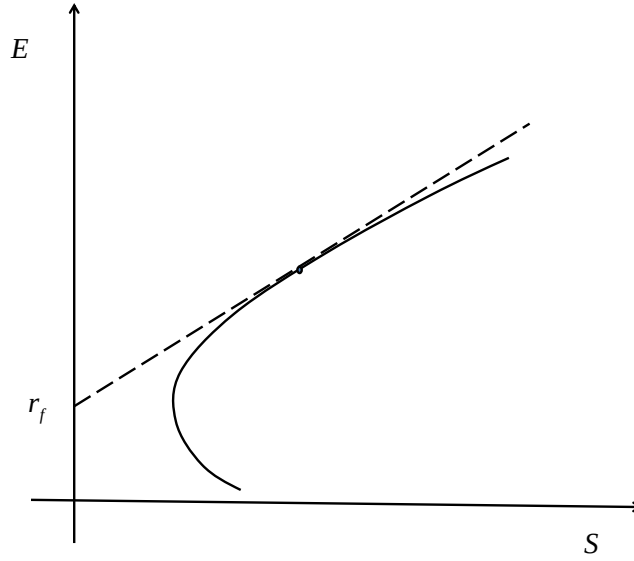


Figure 1: Expected Value – Standard Deviation Frontier with Risk-Free Asset

$$\begin{aligned}
 \sigma^2 &= w^* \iota \Sigma w^* = \frac{(\mu - R)^2}{[C - 2RB + R^2A]^2} [\Sigma^{-1} (\bar{z} - R1)]' \Sigma [\Sigma^{-1} (\bar{z} - R1)] \\
 &= \frac{(\mu - R)^2}{[C - 2RB + R^2A]^2} (\bar{z}' - R1') \Sigma^{-1} \Sigma \Sigma^{-1} (\bar{z} - R1) \\
 &= \frac{(\mu - R)^2}{[C - 2RB + R^2A]^2} [\bar{z}' \Sigma^{-1} \bar{z} - R1' \Sigma^{-1} \bar{z} - R \bar{z}' \Sigma^{-1} 1 + R^2 1' \Sigma^{-1} 1] \\
 &= \frac{(\mu - R)^2}{[C - 2RB + R^2A]^2} [C - 2RB + R^2A] \\
 \sigma^2 &= \frac{(\mu - R)^2}{[C - 2RB + R^2A]}
 \end{aligned} \tag{3}$$

Note

$$\sigma = (\mu - R) \frac{1}{\sqrt{C - 2RB + R^2A}} \quad (4)$$

$\therefore \mu = R + (C - 2RB + R^2A)^{1/2} \sigma \Rightarrow$  CML or the tradeoff between expected return and standard deviation is linear.

C. Next we want to show that the portfolio containing no risk free asset is on the original EV frontier.

1. If the portfolio contains no riskfree asset

$$\begin{aligned} w_0^* &= 1 - 1'w^* = 0 \Rightarrow 1'w_t^* = 1 \\ 1' \left[ \frac{(\mu - R)}{[C - 2RB + R^2A]} \Sigma^{-1} (\bar{z} - R1) \right] &= 1 \\ &= \frac{\mu - R}{C - 2RB + R^2A} [1' \Sigma^{-1} \bar{z} - R1' \Sigma^{-1} 1] = 1 \\ \frac{\mu - R}{C - 2RB + R^2A} [B - RA] &= 1 \\ \Rightarrow \frac{\mu - R}{C - 2RB + R^2A} &= \frac{1}{B - RA} \quad (5) \\ \mu - R &= \frac{C - 2RB + R^2A}{B - RA} \\ \Rightarrow w_t^* &= \frac{\frac{C - 2RB + R^2A}{B - RA}}{C - 2RA - R^2A} \Sigma^{-1} (\bar{z} - R1) \\ &\Rightarrow w_t^* = \frac{\Sigma^{-1} (\bar{z} - R1)}{B - RA} \end{aligned}$$

What is the expected rate of return for that portfolio (containing no riskfree asset)?

$$\begin{aligned}
 \bar{z}_t &= \bar{z}' w_t^* = \bar{z}' \left[ \frac{\Sigma^{-1} (\bar{z} - R1)}{B - RA} \right] \\
 &= \frac{\bar{z}' \Sigma^{-1} \bar{z} - R \bar{z}' \Sigma^{-1} 1}{B - RA} \\
 \bar{z}_t &= \frac{C - RB}{B - RA}.
 \end{aligned} \tag{6}$$

What is the variance for the portfolio containing no riskfree asset?

$$\begin{aligned}
 \sigma_t^2 &= w_t' \Sigma w_t = \left[ \frac{\Sigma^{-1} (\bar{z} - R1)}{B - RA} \right]' \Sigma \left[ \frac{\Sigma^{-1} (\bar{z} - R1)}{B - RA} \right] \\
 &= \frac{(\bar{z}' - R1') \Sigma^{-1} \Sigma \Sigma^{-1} (\bar{z} - R1)}{(B - RA)^2} \\
 &= \frac{\bar{z}' \Sigma^{-1} \bar{z} - R \bar{z}' \Sigma^{-1} 1 - R1' \Sigma^{-1} \bar{z} - R^2 1' \Sigma^{-1} 1}{(B - RA)^2} \\
 &= \frac{C - 2RB + R^2 A}{(B - RA)^2}
 \end{aligned} \tag{7}$$

2. To prove that  $t$  is on the risky asset frontier, we can show that

$$\sigma_t^2 = \frac{A \bar{z}_t^2 - 2B \bar{z}_t + C}{\Delta} = \sigma_p^2 \tag{8}$$

from the risky portfolio frontier.

$$\begin{aligned} \frac{[C - 2RB + R^2A]}{(B - RA)^2} &= \frac{A \left[ \frac{C - RB}{B - RA} \right]^2 - 2B \left[ \frac{C - RB}{B - RA} \right] + C}{\Delta} \\ &= \frac{[C - 2RB + R^2A] \Delta}{(B - RA)^2} = \\ &= \frac{A(C - RA)^2 - 2B(C - RB)(B - RA) + C(B - RA)^2}{(B - RA)^2} \end{aligned} \quad (9)$$

$$\ni: \Delta = AC - B^2$$

$$\begin{aligned} [C - 2RB + R^2A] (AC - B^2) &= A(C^2 - 2RBC + R^2B^2) \\ -2B(CB - RAC - RB^2 + R^2AB) &+ C(B^2 - 2RAB + R^2A^2) \\ 0 &= 0 \end{aligned}$$

## II. Relationships Between Expected Returns

A. The expected excess return on any asset is proportional to its covariance with the expected return on  $z_t$ . As depicted in Figure , excess return  $E[z_i] - R \propto \sigma_{it}$ .

B. Mathematics

1. Let  $\varrho_t$  equal the vector of covariances between  $z_i$  and the portfolio  $w_t$  with expected return  $\bar{z}_t$

$$\begin{aligned} \varrho_t = \Sigma w_t &= \Sigma \left[ \frac{\Sigma^{-1}(\bar{z} - R1)}{B - AR} \right] \\ &= \frac{\bar{z} - R1}{B - AR} \end{aligned} \quad (10)$$

Therefore, computing the variance matrix

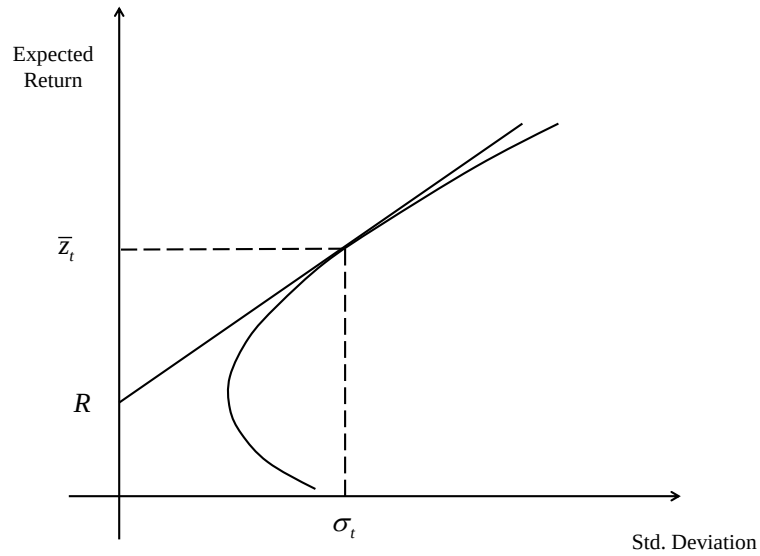


Figure 2: Excess Return

$$\begin{aligned}
 \sigma_t^2 &= w_t' \mathcal{Q}_t = w_t' \left[ \frac{\bar{z} - R1}{B - AR} \right] \\
 &= \frac{w_t' - R w_t' 1}{B - AR} \\
 &= \frac{\bar{z}_t - R}{B - AR} \\
 B - AR &= \frac{\bar{z}_t - R}{\sigma_t^2}
 \end{aligned}
 \tag{11}$$

Finishing the result

$$\begin{aligned}
 \therefore \sigma_t^2 &= \frac{\bar{z} - R1}{B - AR} \\
 \sigma_t^2 [B - AR] &= \bar{z} - R1 \\
 \sigma_t^2 \left[ \frac{\bar{z}_t - R}{\sigma_t^2} \right] &= \bar{z} - R1 \quad (12) \\
 \bar{z} - R1 &= \frac{\sigma_t^2}{\sigma_t^2} [\bar{z}_t - R] \\
 = \beta_t [\bar{z}_t - R] \quad \ni: \quad \beta &\equiv \frac{\sigma_t^2}{\sigma_t^2} \equiv \frac{\text{Cov}(z_t, z^*)}{\text{Var}(z^*)}
 \end{aligned}$$

where  $z^*$  is the return at the point of tangency. After some thought, I have decided that this line is actually the Security Market Line (SML).

2. Other portfolios

a.  $w_d$

$$\begin{aligned}
 w_d &= \frac{\Sigma^{-1} \bar{z}}{B} \\
 \sigma_d &= \Sigma w_d = \Sigma \left[ \frac{\Sigma^{-1} \bar{z}}{B} \right] = \frac{\bar{z}}{B} \\
 \sigma_d^2 &= w_d' \sigma_d = w_d \frac{\bar{z}}{B} = \frac{\bar{z}_d}{B} \\
 B &= \frac{\bar{z}_d}{\sigma_d^2} \quad (13) \\
 \therefore \sigma_d &= \frac{\bar{z}}{\frac{\bar{z}_d}{\sigma_d^2}} \\
 \bar{z} &= \frac{z_d}{\sigma_d^2} \sigma_d
 \end{aligned}$$

b. For a portfolio that is a general combination of  $w_g$  and  $w_d$

$$w_a = (1 - a) w_g + a w_d$$

$$w_g = \frac{\Sigma^{-1}1}{A} \quad w_d = \frac{\Sigma^{-1}\bar{z}}{B}$$

$$\varpi_a = \Sigma w_a = \Sigma \left[ (1 - a) \frac{\Sigma^{-1}1}{A} + a \frac{\Sigma^{-1}\bar{z}}{B} \right]$$

$$= \frac{(1 - a)}{A} 1 + \frac{a}{B} \bar{z}$$

$$\begin{aligned} \sigma_a^2 &= w_a' \Sigma w_a = \left[ \frac{(1 - a)}{A} \Sigma^{-1}1 + \frac{a}{B} \Sigma^{-1}\bar{z} \right]' \left[ \frac{(1 - a)}{A} 1 + \frac{a}{B} \bar{z} \right] \\ &= \frac{(1 - a)^2}{A^2} 1' \Sigma^{-1} 1 + \frac{(1 - a)a}{AB} 1' \Sigma^{-1} \bar{z} + \frac{a(1 - a)}{AB} \bar{z}' \Sigma^{-1} 1 + \frac{a^2}{B^2} \bar{z}' \Sigma^{-1} \bar{z} \\ &= \frac{(1 - a)^2}{A^2} A + \frac{(1 - a)a}{AB} B + \frac{(1 - a)a}{AB} B + \frac{a^2}{B^2} C \\ &= \frac{(1 - a)^2}{A} + \frac{2a(1 - a)}{A} + \frac{a^2}{B^2} C \end{aligned} \tag{14}$$

Turning to the variance of  $a$

$$\begin{aligned} \sigma_a^2 &= w_a' \left[ \frac{(1 - a)}{A} 1 + \frac{a}{B} \bar{z} \right] \\ &= \frac{(1 - a)}{A} w_1' 1 + \frac{a}{B} \bar{z}_t \\ &= \frac{1 - a}{A} + \frac{a}{B} \bar{z}_t \end{aligned} \tag{15}$$

Turning to the covariance between  $p$  and  $a$

$$\begin{aligned}\sigma_{pa} &= w'_p \left[ \frac{(1-a)}{A} 1 + \frac{a}{B} \bar{z} \right] \\ &= \frac{(1-a)}{A} w'_p 1 + \frac{a}{B} w'_p \bar{z} \\ &= \frac{(1-a)}{A} + \frac{a}{B} \bar{z}_p\end{aligned}\tag{16}$$

Working with these two equations yields

$$\begin{aligned}\frac{a}{B} &= \frac{\sigma_a^2 - \sigma_{pa}}{\bar{z}_a - \bar{z}_p} \\ \frac{(1-a)}{A} &= \frac{\bar{z}_a \sigma_{pa} - \bar{z}_p \sigma_a^2}{\bar{z}_a - \bar{z}_p}\end{aligned}\tag{17}$$