

Lecture V: Mean Variance – II - Derivation of the Minimum-Variance Frontier, Minimum-Variance Portfolio, and the Efficient Set with a Risk-Free Asset

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March 20, 2013

I. Derivation of the Minimum-Variance Frontier

A. The minimum-variance frontier is the locus of points that represent the minimum-variance for a given return.

B. Mathematical Derivation

1. General Formulation

$$\begin{aligned}\bar{z} &= w'\bar{z} + w_0R = \sum_{i=0}^N w_i\bar{z}_i \\ \sigma^2 &= w'\Sigma w = \sum_{i=1}^N \sum_{j=1}^N w_i w_j \sigma_{ij}\end{aligned}\tag{1}$$

2. Mean-variance frontier without a risk-free asset (see Lecture 4 for more details)

$$\left. \begin{array}{l} \min \frac{1}{2} w' \Sigma w \\ \text{st } 1' w = 1 \\ \bar{z}' w = \mu \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} w^* = \lambda \Sigma^{-1} 1 + \gamma \Sigma^{-1} \bar{z} \\ \lambda = \frac{C - \mu B}{AC - B^2} \\ \gamma = \frac{\mu - \lambda B}{C} \\ \ni: A \equiv 1' \Sigma^{-1} 1 \quad B \equiv \bar{z}' \Sigma^{-1} 1 \quad C \equiv \bar{z}' \Sigma^{-1} \bar{z} \end{array} \right.\tag{2}$$

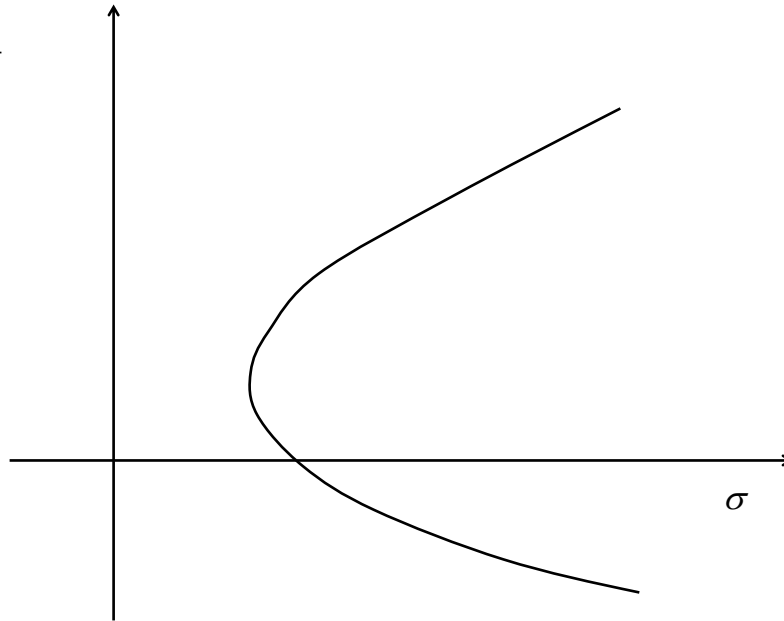


Figure 1: Expected Value – Standard Deviation Frontier

3. Example:

$$\Sigma = \begin{bmatrix} 0.0076 & -0.0008 & -0.0042 \\ -0.0008 & 0.0218 & 0.0035 \\ -0.0042 & 0.0035 & 0.0136 \end{bmatrix} \quad (3)$$

$$\bar{z} = \begin{bmatrix} 0.1355 \\ 0.0735 \\ 0.0017 \end{bmatrix}$$

Therefore $A = 1'\Sigma^{-1}1 = 364.5317$, $B = \bar{z}'\Sigma^{-1}1 = 30.2867$,
and $C = \bar{z}'\Sigma^{-1}\bar{z} = 3.1256$. Taking $\mu = 0.05$ yields

$$\begin{aligned}\lambda &= 0.0073 \\ \gamma &= -0.0543 \\ w^* &= \begin{bmatrix} 0.3266 \\ 0.0639 \\ 0.6095 \end{bmatrix} \end{aligned} \quad (4)$$

$$\begin{aligned}z'w &= 0.05 \\ w'\Sigma w &= 0.0045\end{aligned}$$

Duplicate for $\mu = 0$ to 0.25.

II. Minimum Variance Portfolio

- A. We can define the minimum-variance portfolio as the lowest possible variance portfolio attainable from a given set of assets as depicted graphically in Figure 2

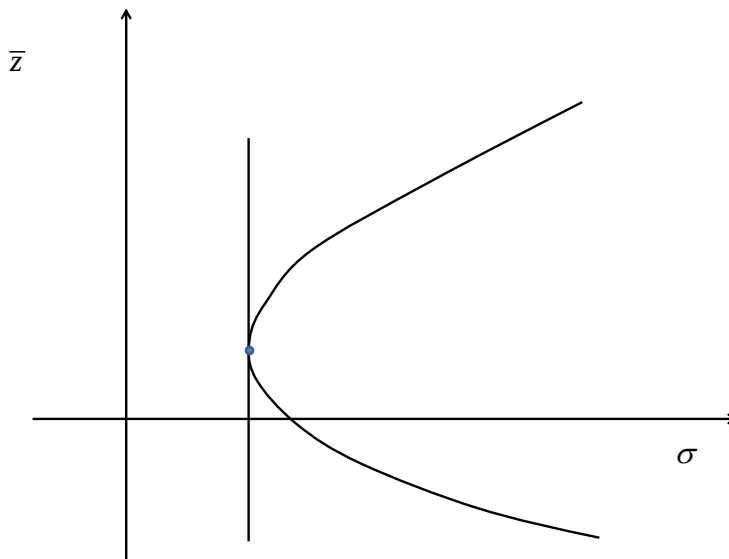


Figure 2: Minimum Variance Portfolio

B. Mathematical Derivation

$$\begin{aligned}
 w' \Sigma w &= w' \Sigma (\lambda \Sigma^{-1} 1 + \gamma \Sigma^{-1} \bar{z}) \\
 &= \lambda w' \Sigma \Sigma^{-1} 1 + \gamma w' \Sigma \Sigma^{-1} \bar{z} \\
 &= \lambda w' 1 + \gamma w' \bar{z} \\
 &= \lambda 1 + \gamma \mu \\
 &= \frac{C - \mu B}{AC - B^2} + \frac{\mu A - B}{AC - B^2} \mu
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 \sigma_p^2(\mu) &= \frac{C - \mu B + \mu^2 A - B\mu}{AC - B^2} \\
 &= \frac{A\mu^2 - 2B\mu + C}{AC - B^2}
 \end{aligned}$$

To find the minimum

$$\begin{aligned}
 \frac{d\sigma_p^2(\mu)}{d\mu} &= \frac{2A\mu - 2B}{AC - B^2} = 0 \\
 \Rightarrow 2A\mu - 2B &= 0 \Rightarrow \mu^* = \frac{B}{A} \\
 \Rightarrow \sigma_p^2(\mu^*) &= \frac{(B/A)^2 A - 2(B/A) B + C}{AC - B^2} \\
 &= \frac{\frac{B^2}{A} - \frac{2B^2}{A} + C}{AC - B^2} \\
 &= \frac{\frac{-B^2 + AC}{A}}{AC - B^2} = \frac{1}{A}
 \end{aligned} \tag{6}$$

In this example

$$\mu = \frac{30.2867}{364.5317} = 0.08308$$

$$\sigma^2 = 0.0027$$

$$\lambda = \frac{C - \mu B}{AC - B^2} = -0.0011$$

(7)

$$\gamma = \frac{0.08308 \times 364.5317 - 30.2867}{364.5317 \times 3.1256 - 30.2867^2} = -6.3322E - 06 \approx 0$$

$$w^* = \lambda \Sigma^{-1} \mathbf{1} = \lambda \begin{bmatrix} 206.1928 \\ 32.7634 \\ 128.7748 \end{bmatrix}$$

C. Slopes of Asymptotes - Depicted in Figure 3

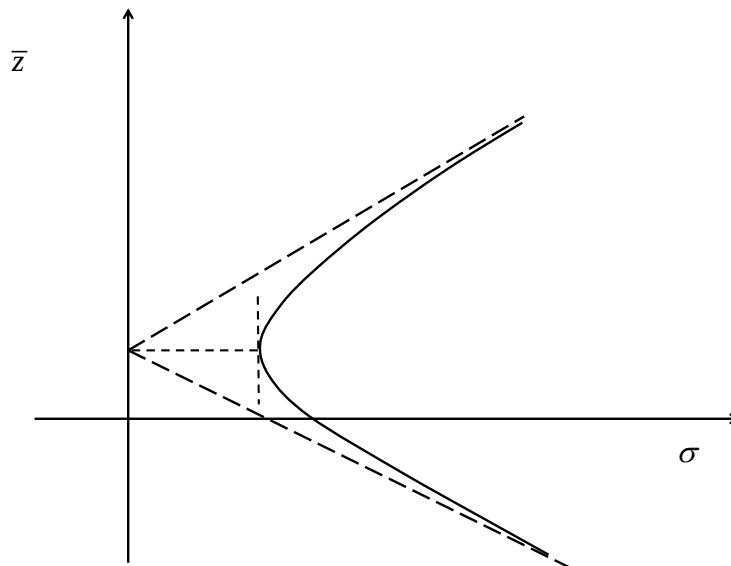


Figure 3: Asymptotes

$$\frac{d\mu}{d\sigma} = \frac{d\mu}{d\sigma^2} \frac{d\sigma^2}{d\sigma} = \frac{\Delta}{2A\mu - 2B} 2\sigma \quad (8)$$

where $\Delta = AC - B^2$ is some constant. Note that this is the result of Green's Theorem. From Equation 5

$$\begin{aligned} \frac{d\mu}{d\sigma^2} &= \frac{1}{\frac{d\sigma^2}{d\mu}} \\ &= \frac{1}{\frac{2A\mu - 2B}{AC - B^2}} \\ &\Rightarrow \frac{AC - B^2}{2A\mu - 2B} \end{aligned} \quad (9)$$

Again, based on the results from Equation 5

$$\begin{aligned} \sigma = \sqrt{\sigma^2} &= \sqrt{\frac{\mu^2 A - 2\mu B + C}{\Delta}} \\ &= \frac{\sqrt{\mu^2 A - 2\mu B + C}}{\sqrt{\Delta}} \end{aligned} \quad (10)$$

Therefore

$$\begin{aligned} \frac{d\mu}{d\sigma} &= \frac{\Delta}{A\mu - B} \frac{\sqrt{\mu^2 A - 2\mu B + C}}{\sqrt{\Delta}} \\ &= \frac{\sqrt{\Delta}}{A\mu - B} \sqrt{\mu^2 A - 2\mu B + C} \end{aligned} \quad (11)$$

Taking the limit

$$\lim_{\mu \rightarrow \infty} \frac{d\mu}{d\sigma} = \lim_{\mu \rightarrow \infty} \sqrt{\Delta} \frac{\sqrt{A\mu^2 - 2B\mu + C}}{A\mu - B} \quad (12)$$

Focusing on the $A\mu^2$ term because of its magnitude

$$\frac{\sqrt{A\mu^2}}{A\mu} \Rightarrow \frac{\sqrt{A}\mu}{A\mu} \Rightarrow \frac{1}{\sqrt{A}} \quad (13)$$

Thus the limit becomes

$$\lim_{\mu \rightarrow \infty} \frac{d\mu}{d\sigma} = \pm \sqrt{\frac{\Delta}{A}} \quad (14)$$

D. Any point on the frontier can be expressed as a linear combination of two other points on the frontier.

1. From

$$w^* = \lambda \Sigma^{-1} 1 + \gamma \Sigma^{-1} \bar{z} \quad (15)$$

In the minimum variance portfolio $\gamma = 0 \Rightarrow w^* = \lambda \Sigma^{-1} 1$.

2. For the second portfolio use

$$w_d \equiv \frac{\Sigma^{-1} \bar{z}}{B} \quad (16)$$

3. $w^* = (\lambda A) w_g + (\gamma B) w_d$ where $w_g = \Sigma^{-1} 1 / A$ and $w_d = \Sigma^{-1} \bar{z} / B$ yielding $w^* = \lambda \Sigma^{-1} 1 + \gamma \Sigma^{-1} \bar{z}$. Since $\lambda A + \gamma B = 1$, any w^* can be written as a linear function of w_g and w_d

$$\left[\frac{C - \mu B}{\Delta} \right] A + \left[\frac{\mu A - B}{\Delta} \right] B = \frac{AC - \mu AB + \mu AB - B^2}{\Delta} = 1 \quad (17)$$

E. In addition

$$\text{Cov}(z_g, z_p) = w_g' \Sigma w_p$$

$$w_g = \frac{\Sigma^{-1} 1}{A} \Rightarrow w_g' = \frac{1' \Sigma^{-1}}{A} \quad (18)$$

$$\therefore \text{Cov}(z_g, z_p) = \frac{1' \Sigma^{-1}}{A} \Sigma w_p = \frac{1' w_p}{A} = \frac{1}{A}$$

III. Efficient Set with a Riskfree Asset

$$\min \frac{1}{2} w' \Sigma w \quad (19)$$

$$\text{st } (\bar{z} - R1)' w = \mu - R$$

Forming the Lagrangian

$$L \equiv \frac{1}{2} w' \Sigma w + \gamma (\mu - R - (\bar{z} - R1)' w)$$

$$\nabla_w L = \Sigma w - \gamma (\bar{z} - R1) = 0 \quad (20)$$

$$w^* = \gamma \Sigma^{-1} (\bar{z} - R1)$$

$$w_0^* = 1 - 1' w^* = 1$$

Substituting the result for w^* back into the constraint yields

$$(\bar{z} - R1)' [\gamma \Sigma^{-1} (\bar{z} - R1)] = \mu - R$$

$$\gamma [\bar{z}' \Sigma^{-1} \bar{z} - R \bar{z}' \Sigma^{-1} 1 - R1' \Sigma^{-1} \bar{z} + R^2 1' \Sigma^{-1} 1] = \mu - R$$

$$\gamma [C - 2RB - R^2 A] = \mu - R \quad (21)$$

$$\gamma = \frac{\mu - R}{C - 2RB + R^2 A}$$

$$w^* = \frac{\mu - R}{C - 2RB + R^2 A} \Sigma^{-1} (\bar{z} - R1)$$