

Lecture IV: Mean Variance – I

Charles B. Moss

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I. Measurement of Risk and Return under Normality.

A. Basic Problem

1. At the outset it is useful to define what we mean by return.
 - a. In Copeland and Weston's Bayside Cigar example, they define returns as

$$R = \frac{W - I}{I} \quad (1)$$

where R is the rate of return on investment, W is the terminal wealth and I is the initial investment. The argument is that

$$\begin{aligned} W &= (1 + R) I, \text{ or} \\ I &= \frac{W}{1 + R}. \end{aligned} \quad (2)$$

- b. However, dealing with stock markets necessitates a slight change in the formulation. Specifically,

$$R = \frac{(P_1 + D_1) - P_0}{P_0} \quad (3)$$

where P_i is the price of stock in period i and D_i is the dividend distributed during period i .

- c. Finally, an alternative formulation is

$$R = \ln(P_1 + D_1) - \ln(P_0) \quad (4)$$

which expresses the same concept in continuous time.

2. Copeland and Weston then proceed to list three measures of location

- a. The mean

$$E[\tilde{x}] = \sum_{i=1}^N p_i x_i. \quad (5)$$

where x_i is an observed value of the random variable, p_i is the probability associated with that observation, and N is the sample size.

- b. The median – the value of $\tilde{x}$ where $P[x_i \leq x_{\text{Median}}] = 0.50$ and $P[x_i > x_{\text{Median}}] = 0.50$.
c. The mode – the most likely outcome of $\tilde{x}$.
d. In defining the mean they present two useful properties

$$\begin{aligned} \text{(I.) } E[\tilde{x} + a] &= E[\tilde{x}] + a \\ \text{(II.) } E[a\tilde{x}] &= aE[\tilde{x}] \end{aligned} \quad (6)$$

3. Measures of Dispersion

- a. Range = $R_{\text{max}} - R_{\text{min}}$.
b. Semi-interquartile Range

$$= \frac{x|_{P[\tilde{x} < x]=0.75} - x|_{P[\tilde{x} > x]=0.75}}{2} \quad (7)$$

- c. Variance

$$\begin{aligned} V(x) &= \sum_{i=1}^N p_i [x_i - E[\tilde{x}]]^2 \\ \sigma_x &= \sqrt{V[x]} \end{aligned} \quad (8)$$

Properties

- $V[\tilde{x} + a] = V[x]$. An alternative formulation

$$\sigma_x^2 = \sum_{i=1}^N p_i x_i^2 - E[\tilde{x}]^2 \quad (9)$$

let $\hat{x} = x + a$

$$\begin{aligned}
 & \sum_{i=1}^N p_i (x_i + a)^2 - E[\tilde{x} + a]^2 \\
 &= \sum_{i=1}^N p_i (x_i^2 + 2x_i a + a^2) - (E[x] + a)^2 \\
 &= \sum_{i=1}^N p_i x_i^2 + 2a \sum_{i=1}^N p_i x_i + a^2 \sum_{i=1}^N p_i - (E[x]^2 + 2E[x]a + a^2) \\
 &= \left(\sum_{i=1}^N p_i x_i^2 - E[x]^2 \right) + (2aE[x] - 2aE[x]) + (a^2 - a^2) \\
 &= \sigma_x^2
 \end{aligned} \tag{10}$$

· $V[ax] = a^2 V[x]$, let $\hat{x} = ax$

$$\begin{aligned}
 & \sum_{i=1}^N p_i (ax_i^2)^2 - E[a\tilde{x}]^2 \\
 &= \sum_{i=1}^N p_i a^2 x_i^2 - [aE[\tilde{x}]]^2 \\
 &= a^2 \sum_{i=1}^N p_i x_i^2 - a^2 E[x]^2 \\
 &= a^2 \left[\sum_{i=1}^N p_i x_i^2 - E[x]^2 \right] = a^2 V[x]
 \end{aligned} \tag{11}$$

d. Semivariance

$$\begin{aligned}
 \text{SEMIVAR} &= \sum_{i=1}^N \delta(x_j, E(x)) p_i (x_i - E[x])^2 \\
 \delta(x_j, E(x)) &= \begin{cases} 1 & \text{if } x_j < E[x] \\ 0 & \text{if } x_j \geq E[x] \end{cases}
 \end{aligned} \tag{12}$$

e. Mean Absolute Deviation (MAD)

$$\text{MAD} = \sum_{i=1}^N p_i [|x_i - E[x]|] \quad (13)$$

B. The normal distribution

1. The probability density function for the normal distribution is

$$f(x; \mu, \sigma^2) = \frac{1}{(2\pi)^{1/2} \sigma} \exp \left[-\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2} \right] \quad (14)$$

2. The moment generating function for the normal distribution is

$$\begin{aligned} \Psi(t) = E(\exp(tx)) &= \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2} \sigma} \exp \left[tx - \frac{(x - \mu)^2}{2\sigma^2} \right] dx \\ &= \exp \left[\mu t + \frac{1}{2} \sigma^2 t^2 \right] \end{aligned} \quad (15)$$

3. The univariate normal has two sufficient statistics

$$\begin{aligned} T_1 &= \sum_{i=1}^N x_i \\ T_2 &= \sum_{i=1}^N x_i^2 \end{aligned} \quad (16)$$

4. Sometimes we transform the probability density function to the standard normal

$$z = \frac{x - \mu}{\sigma} \Rightarrow f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} \exp \left[-\frac{z^2}{2} \right] \quad (17)$$

5. Multivariate Normal

- a. Now we are interested in a vector of normal variables

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (18)$$

- b. The multivariate normal is characterized by two sets of coefficients.
- The mean vector

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad (19)$$

- The covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \quad (20)$$

- c. The probability density function is then

$$f(x; \mu, \Sigma) = \frac{1}{\sqrt{2\pi}} |\Sigma|^{-\frac{1}{2}} \exp \left[-\frac{1}{2} (x - \mu)' \Sigma^{-1} (x - \mu) \right] \quad (21)$$

- d. The μ and Σ are typically estimated by

$$\hat{\mu} = \frac{1}{N} \begin{bmatrix} \sum_{i=1}^N x_{i1} \\ \sum_{i=1}^N x_{i2} \end{bmatrix} \text{ or } \mu = \frac{1}{N} x'1 \quad (22)$$

where 1 is a vector of ones $N \times 1$. The matrix of variances are then estimated by

$$\begin{aligned} \hat{\Sigma} &= \frac{1}{N} x'x - \hat{\mu}\hat{\mu}' \\ &= \frac{1}{N} \begin{bmatrix} \sum_{i=1}^N x_{i1}x_{i1} & \sum_{i=1}^N x_{i1}x_{i2} \\ \sum_{i=1}^N x_{i2}x_{i1} & \sum_{i=1}^N x_{i2}x_{i2} \end{bmatrix} - \begin{bmatrix} \hat{\mu}_1\hat{\mu}_1 & \hat{\mu}_1\hat{\mu}_2 \\ \hat{\mu}_2\hat{\mu}_1 & \hat{\mu}_2\hat{\mu}_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{N} \sum_{i=1}^N x_{i1}x_{i1} - \hat{\mu}_1\hat{\mu}_1 & \frac{1}{N} \sum_{i=1}^N x_{i1}x_{i2} - \hat{\mu}_1\hat{\mu}_2 \\ \frac{1}{N} \sum_{i=1}^N x_{i2}x_{i1} - \hat{\mu}_2\hat{\mu}_1 & \frac{1}{N} \sum_{i=1}^N x_{i2}x_{i2} - \hat{\mu}_2\hat{\mu}_2 \end{bmatrix} \end{aligned} \quad (23)$$

- Notice that the diagonals are identical to the definition of the variance in the univariate case.
- To explain the off-diagonals, first notice that they are identical – the variance matrix is symmetric. To explain this further let us begin with the expression

$$\begin{aligned}
 \text{Cov}[x_1, x_2] &= E[(x_1 - \mu_1)(x_2 - \mu_2)] \\
 &= E[x_1x_2 - x_1\mu_1 - x_2\mu_2 + \mu_1\mu_2] \\
 &= E[x_1x_2] - E[x_1]\mu_1 - E[x_2]\mu_2 + \mu_1\mu_2 \\
 &\Rightarrow \frac{1}{N} \sum_{i=1}^N x_{i1}x_{i2} - \hat{\mu}_1\hat{\mu}_2
 \end{aligned} \tag{24}$$

Thus, the covariance terms measure the tendency for the returns on both assets to be high or low at the same time.

e. Defining correlation

$$\rho_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} \tag{25}$$

Consider it a normalized measure of covariance. Alternatively, it is sometimes useful to use

$$\sigma_{12} = \sqrt{\sigma_{11}\sigma_{22}}\rho_{12}. \tag{26}$$

II. Multivariate Normality - Risk and Return on a Portfolio

A. Often in finance we want to create a new asset by combining two or more existing assets. For example

$$z = ax_1 + (1 - a)x_2 \tag{27}$$

1. Basically this amounts to forming a new random variable that is the sum of two other random variables. In this cases

$$z = \begin{bmatrix} a & (1-a) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{28}$$

2. Under multivariate normality, the mean of this new distribution becomes

$$\mu_z = \begin{bmatrix} a & (1-a) \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = a\mu_1 + (1-a)\mu_2 \tag{29}$$

And the variance becomes

$$\begin{aligned}
 \sigma_z^2 &= \begin{bmatrix} a & (1-a) \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} a \\ (1-a) \end{bmatrix} \\
 &= \begin{bmatrix} a\sigma_{11} + (1-a)\sigma_{21} & a\sigma_{12} + (1-a)\sigma_{22} \end{bmatrix} \begin{bmatrix} a \\ (1-a) \end{bmatrix} \\
 &= a^2\sigma_{11} + (1-a)a\sigma_{21} + a(1-a)\sigma_{12} + (1-a)^2\sigma_{22} \\
 &\quad \text{since } \sigma_{21} = \sigma_{12} \\
 &= a^2\sigma_{11} + (1-a)^2\sigma_{22} + 2a(1-a)\sigma_{12}
 \end{aligned} \tag{30}$$

3. Example: Suppose that the expected rate of return on a utility stock was 0.02 with a variance of 0.0004. In addition, assume that the rate of return on a junk bond was 0.15 with a variance of 0.05. Imagine the following portfolios

Table 1: Return on Two Asset Portfolio						
Mean	<i>a</i>	Correlation Coefficient ρ_{12}				
Return		1.0	0.5	0.0	-0.5	-1.0
0.020	1.00	0.0004	0.0004	0.0004	0.0004	0.0004
0.053	0.75	0.0050	0.0042	0.0034	0.0025	0.0017
0.085	0.50	0.0148	0.0137	0.0126	0.0115	0.0104
0.118	0.25	0.0298	0.0290	0.0282	0.0273	0.0265
0.150	1.00	0.0500	0.0500	0.0500	0.0500	0.0500

Notice that as $\rho \rightarrow 1$ the frontier becomes linear as depicted in Figure 1.

4. Formally, how does the trade-off between expected returns and variance change as a changes?

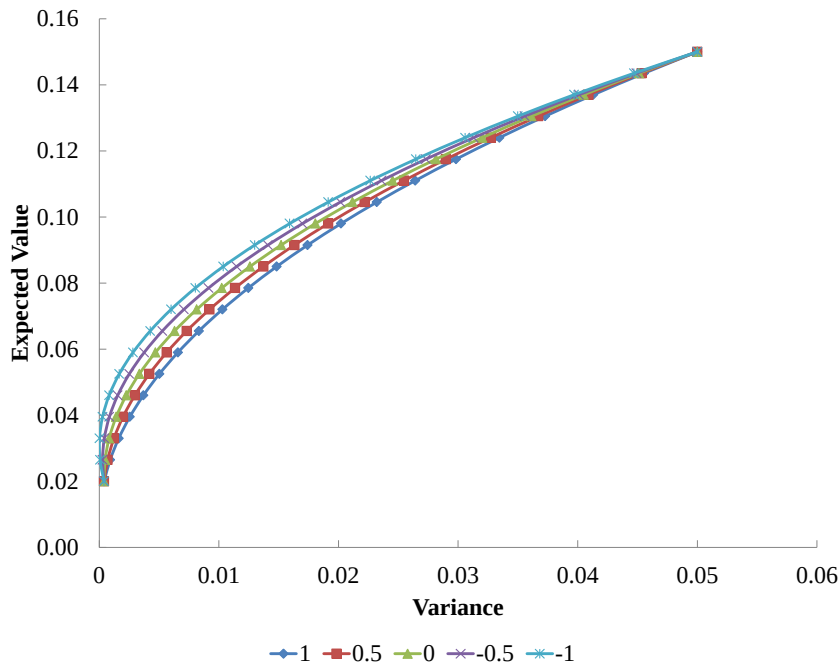


Figure 1: Expected Value – Variance Frontier

$$\begin{aligned}
 \frac{d\mu_p}{da} &= \frac{d}{da} [a\mu_1 + (1-a)\mu_2] \\
 &= \mu_1 - \mu_2 \\
 \frac{d\sigma_p^2}{da} &= \frac{d}{da} [a^2\sigma_{11} + (1-a)^2\sigma_{22} + 2a(1-a)\sigma_{12}] \\
 &= \frac{d}{da} [a^2\sigma_{11} + (1-2a+a^2)\sigma_{22} + (2a-2a^2)\sigma_{12}] \\
 &= 2a\sigma_{11} + (-2+2a)\sigma_{22} + (2-4a)\sigma_{12} \\
 &= 2a\sigma_{11} - 2\sigma_{22} + 2a\sigma_{22} + 2\sigma_{12} - 4a\sigma_{12} \\
 &= a(2\sigma_{11} + 2\sigma_{22} - 4\sigma_{12}) - 2\sigma_{22} + 2\sigma_{12} \\
 &= 2a(\sigma_{11} + \sigma_{22} - 2\sigma_{12}) + 2(\sigma_{12} - \sigma_{22})
 \end{aligned} \tag{31}$$

$$\therefore \frac{d\mu_p}{d\sigma_p^2} = \frac{\frac{d\mu_p}{da}}{\frac{d\sigma_p^2}{da}} = \frac{\mu_1 - \mu_2}{2a(\sigma_{11} + \sigma_{22} - 2\sigma_{12}) + 2(\sigma_{12} - \sigma_{22})} \quad (32)$$

If $\rho_{12} = 1$

$$\begin{aligned} \sigma_p^2 &= a^2 \sigma_{11} + 2a(1-a)\sigma_{11}^{1/2}\sigma_{22}^{1/2} + (1-a)^2 \sigma_{22} \\ &= \left[a\sigma_{11}^{1/2} + (1-a)\sigma_{22}^{1/2} \right]^2 \\ \sigma_p &= a\sigma_{11}^{1/2} + (1-a)\sigma_{22}^{1/2} \\ \frac{d\sigma_p}{da} &= \sigma_{11}^{1/2} - \sigma_{22}^{1/2} \end{aligned} \quad (33)$$

$$\therefore \frac{\frac{d\mu}{da}}{\frac{d\sigma_p}{da}} = \frac{\mu_1 - \mu_2}{\sigma_{11}^{1/2} - \sigma_{22}^{1/2}} = \text{constant}$$

b. E-S Frontier with 2 assets

1. The E-S frontier with two assets is bounded between two extremes (i.e., two curves between A and B) determined by $-1 \geq \rho \geq 1$ as depicted in Figure 2

$$\mu_p = a\mu_1 + (1-a)\mu_2 \quad (34)$$

2. In the case of more than two assets their can be points on the inside of the parabola as depicted in Figure 3

III. Efficient Portfolio – The Mean-Variance Frontier

A. From Ingersoll – Assuming a utility function $V[\bar{Z}, \sigma^2]$ where $V_1 > 0$ and $V_2 < 0$. Under these assumptions the optimal portfolios will be those portfolios that yield the greatest expected return for a given level of variance. These portfolios are called mean-variance efficient.

1. Mean-variance efficient portfolios are a subset of the minimum variance portfolios that include the portfolio with the smallest variance possible for any possible level of expected income.

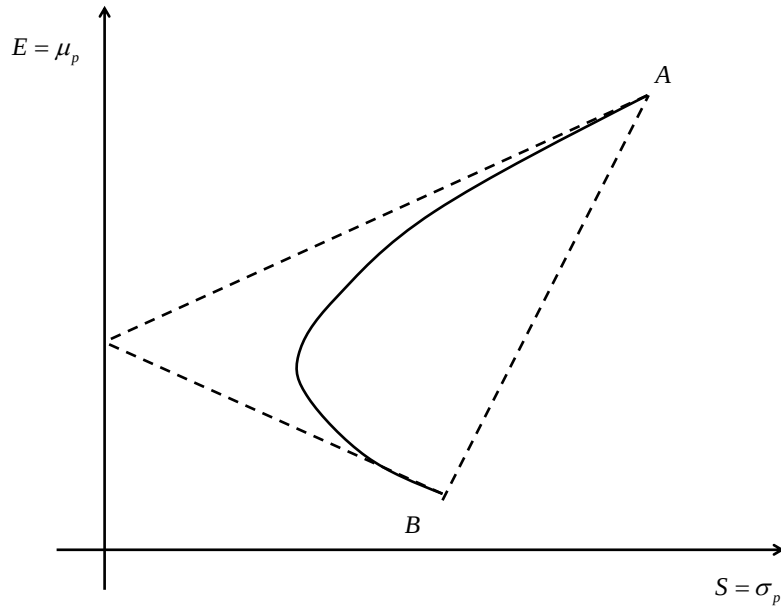


Figure 2: E-S frontier

$$\begin{aligned}
 & \min \frac{1}{2} w' \Sigma w \\
 & \text{st } 1' w = 1 \\
 & \quad \bar{z}' w = \mu \\
 \Rightarrow L &= \frac{w' \Sigma w}{2} + \lambda (1 - 1' w) + \gamma (\mu - \bar{z}' w) \\
 \nabla_w L &= \Sigma w - \lambda 1 - \gamma \bar{z} = 0 \\
 w^* &= \lambda \Sigma^{-1} 1 + \gamma \Sigma^{-1} \bar{z}.
 \end{aligned} \tag{35}$$

Remembering

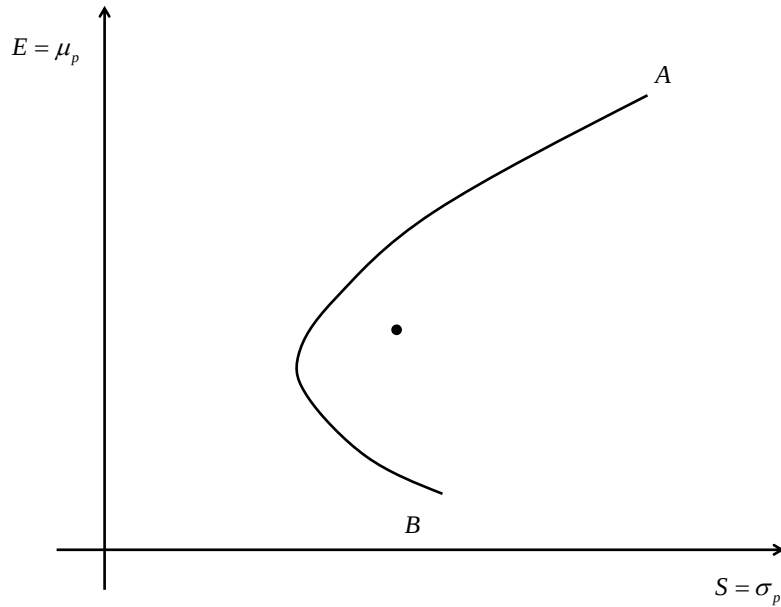


Figure 3: More than Two Assets

$$\begin{aligned}
 w' \Sigma w &= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\
 &= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13} \\ w_1 \sigma_{21} + w_2 \sigma_{22} + w_3 \sigma_{23} \\ w_1 \sigma_{31} + w_2 \sigma_{32} + w_3 \sigma_{33} \end{bmatrix} \quad (36) \\
 &= w_1 (w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13}) + \\
 &\quad w_2 (w_1 \sigma_{21} + w_2 \sigma_{22} + w_3 \sigma_{23}) + \\
 &\quad w_3 (w_1 \sigma_{31} + w_2 \sigma_{32} + w_3 \sigma_{33})
 \end{aligned}$$

Using matrix notation

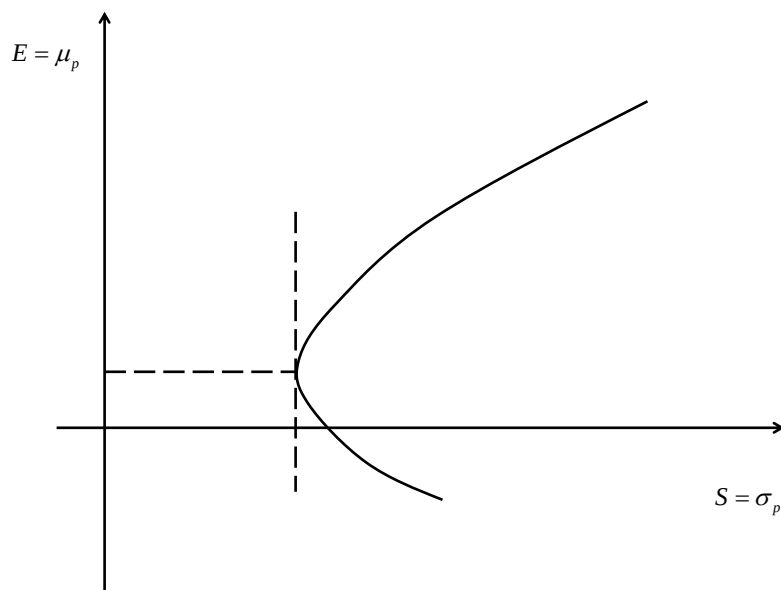


Figure 4: Minimum Variance Portfolio

$$\begin{aligned}
 \nabla_w w' \Sigma w &= \begin{bmatrix} (w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13}) + w_1 \sigma_{11} + w_2 \sigma_{21} + w_3 \sigma_{31} \\ w_1 \sigma_{12} + (w_1 \sigma_{21} + w_2 \sigma_{22} + w_3 \sigma_{23}) + w_2 \sigma_{22} + w_3 \sigma_{32} \\ w_1 \sigma_{13} + w_2 \sigma_{23} + (w_1 \sigma_{31} + w_2 \sigma_{32} + w_3 \sigma_{33}) + w_3 \sigma_{33} \end{bmatrix} \\
 &= \begin{bmatrix} 2(w_1 \sigma_{11} + w_2 \sigma_{12} + w_3 \sigma_{13}) \\ 2(w_1 \sigma_{12} + w_2 \sigma_{22} + w_3 \sigma_{23}) \\ 2(w_1 \sigma_{13} + w_2 \sigma_{23} + w_3 \sigma_{33}) \end{bmatrix} \\
 &= 2 \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}
 \end{aligned} \tag{37}$$

Moving to the portfolio constraint

$$1'w = w_1 + w_2 + w_3$$

$$\nabla_w 1'w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \nabla_w \lambda 1'w = \lambda 1. \quad (38)$$

Similarly,

$$\nabla_w \gamma \bar{z}'w = \nabla_w \gamma [\bar{z}_1 w_1 + \bar{z}_2 w_2 + \bar{z}_3 w_3] = \gamma \bar{z} \quad (39)$$

Bringing the parts together

$$\begin{aligned} \nabla_w L &= \Sigma w - \lambda 1 - \gamma \bar{z} = 0 \\ \Rightarrow w &= \lambda \Sigma^{-1} 1 + \gamma \Sigma^{-1} \bar{z} \end{aligned} \quad (40)$$

using $\bar{z}'w = \mu$

$$\begin{aligned} \bar{z}'(\lambda \Sigma^{-1} 1 + \gamma \Sigma^{-1} \bar{z}) &= \mu \\ \lambda \bar{z}'\Sigma^{-1} 1 + \gamma \bar{z}'\Sigma^{-1} \bar{z} &= \mu \end{aligned} \quad (41)$$

Letting $1'\Sigma^{-1}1 \equiv A$, $\bar{z}'\Sigma^{-1}1 \equiv B$ and $\bar{z}'\Sigma^{-1}\bar{z} \equiv C > 0$

$$\begin{aligned} \lambda B + \gamma C &= \mu \\ \lambda &= \frac{\mu - \gamma C}{B} \end{aligned} \quad (42)$$

Substituting this result back into Equation 40 yields

$$\begin{aligned} w^* &= \frac{\mu - \gamma C}{B} \Sigma^{-1} 1 + \gamma \Sigma^{-1} \bar{z} \\ 1'w &= 1' \left[\frac{\mu - \gamma C}{B} \Sigma^{-1} 1 + \gamma \Sigma^{-1} \bar{z} \right] = 1 \end{aligned} \quad (43)$$

$$\begin{aligned}
 &\Rightarrow \frac{\mu - \gamma C}{B} 1' \Sigma^{-1} 1 + \gamma 1' \Sigma^{-1} \bar{z} = 1 \\
 &\quad \frac{\mu - \gamma C}{B} A + \gamma B = 1 \\
 &\quad \frac{\mu A}{B} - \frac{\gamma C A}{B} + \gamma B = 1 \\
 &\quad \frac{\mu A}{B} - \frac{\gamma C A}{B} + \frac{\gamma B^2}{B} = 1 \\
 &\quad \frac{\mu A}{B} - \frac{\gamma}{B} (AC - B^2) = 1 \\
 &\quad \mu A - \gamma (AC - B^2) = B \\
 &\quad -\gamma (AC - B^2) = B - \mu A \\
 &\quad \gamma = \frac{\mu A - B}{AC - B^2}
 \end{aligned} \tag{44}$$

Focusing back on λ

$$\begin{aligned}
 \lambda &= \frac{\mu - \left(\frac{\mu A - B}{AC - B^2} \right) C}{B} \\
 &= \frac{1}{B} \left[\mu - \left(\frac{\mu A - B}{AC - B^2} \right) C \right] \\
 &= \frac{1}{B} \left[\frac{\mu AC - \mu B^2 - \mu AC + BC}{AC - B^2} \right] \\
 \lambda &= \frac{C - \mu B}{AC - B^2}
 \end{aligned} \tag{45}$$