

Lecture III: Specific Arrow-Pratt, Orange Tree Example, Stochastic Dominance, Beginning Mean-Variance

Charles B. Moss

February 1, 2013

I. An Examination of the Arrow-Pratt coefficients for particular functions.

A. Quadratic Utility: To specify the appropriate shape of the utility function, the quadratic utility function becomes

$$\begin{aligned}U(W) &= aW - bW^2 \\U'(W) &= a - 2bW \\U''(W) &= -2b\end{aligned}\tag{1}$$

Absolute risk aversion

$$\begin{aligned}R_A(W) &= -\frac{-2b}{a - 2bW} \\&= \frac{2b}{a - 2bW}\end{aligned}\tag{2}$$

$$\frac{dR_A(W)}{dW} = 2b \frac{2b}{(a - 2bW)^2} > 0$$

Relative risk aversion

$$\begin{aligned}R_R(W) &= \frac{2bW}{a - 2bW} \\ \frac{dR_R(W)}{dW} &= 2b \frac{2bW}{(a - 2bW)^2} + \frac{2b}{a - 2bW} > 0\end{aligned}\tag{3}$$

B. Power utility function

$$\begin{aligned}
 U(W) &= \frac{W^{1-r}}{1-r} \\
 U'(W) &= W^{-r} \\
 U''(W) &= -rW^{-r-1} \\
 R_A(W) &= - \left[\frac{-rW^{-r-1}}{W^{-r}} \right] = \frac{r}{W} \Rightarrow \frac{dR_A(W)}{dW} < 0 \\
 R_R(W) &= - \left[\frac{rW^{-r-1}W}{W^{-r}} \right] = r
 \end{aligned} \tag{4}$$

Constant relative risk aversion.

C. Negative exponential

$$\begin{aligned}
 U(W) &= -\exp[-\rho W] \\
 U'(W) &= \rho \exp[-\rho W] \\
 U''(W) &= -\rho^2 \exp[-\rho W] \\
 R_A(W) &= - \left[\frac{-\rho^2 \exp[-\rho W]}{\rho \exp[-\rho W]} \right] = \rho \\
 R_R(W) &= - \left[\frac{-\rho^2 \exp[-\rho W] W}{\rho \exp[-\rho W]} \right] = \rho W \Rightarrow \frac{dR_R(W)}{dW} = \rho > 0
 \end{aligned} \tag{5}$$

Constant absolute risk aversion.

D. HARA - Hyperbolic Absolute Risk Aversion

$$\begin{aligned}
 U(W) &= \frac{1-\gamma}{\gamma} \left(\frac{aW}{1-\gamma} + b \right)^\gamma, b > 0 \\
 U'(W) &= (1-\gamma) \left(\frac{aW}{1-\gamma} + b \right)^{\gamma-1} \left(\frac{a}{1-\gamma} \right) \\
 &= a \left(\frac{aW}{1-\gamma} + b \right)^{\gamma-1} \\
 U'' &= a(\gamma-1) \left(\frac{aW}{1-\gamma} + b \right)^{\gamma-2} \left(\frac{a}{1-\gamma} \right) \\
 &= -a^2 \left(\frac{aW}{1-\gamma} + b \right)^{\gamma-2} \\
 R_A(W) &= - \left[\frac{-a^2 \left(\frac{aW}{1-\gamma} + b \right)^{\gamma-2}}{a \left(\frac{aW}{1-\gamma} + b \right)^{\gamma-1}} \right] \\
 &= \frac{a}{\left(\frac{aW}{1-\gamma} + b \right)} \\
 &= \frac{a}{\frac{aW}{1-\gamma} + \frac{b(1-\gamma)}{1-\gamma}} \\
 &= \frac{a}{\frac{aW + b(1-\gamma)}{1-\gamma}} \\
 &= \frac{a(1-\gamma)}{aW + b(1-\gamma)}
 \end{aligned} \tag{6}$$

1. As $\gamma \rightarrow 1$ the decision maker becomes risk neutral (the numerator of the fraction goes to zero).
2. $\gamma = 2$ yields a risk aversion equal to the quadratic utility function.
3. $\gamma \rightarrow -\infty$ and $b = 1$ yields a risk aversion equal to the negative exponential.

4. $b = 0$ and $\gamma < 1$ yields the risk aversion of the power utility function.

II. Orange Tree Example

A. Growing orange trees in Florida can be a fairly risky venture. Sources of risk:

1. One primary source of risk involves the price received for the orange.
 - a. Traditionally, Florida's orange production has primarily been at the mercy of domestic factors such as aggregate income.
 - b. However, with the evolution of the Brazilian citrus crop, other factors such as exchange rates and macroeconomic forces abroad have become important.
2. Another source of uncertainty is the weather.
 - a. In Florida we have the possibility of weather damage primarily from freezes. There are several intensities of freeze:
 - A short freeze with low intensity (i.e., freezing weather for 4 - 8 hours). This type of freeze typically results in fruit damage, but it does not kill mature trees because of their canopy.
 - A killing freeze - this freeze results in various degrees of wood damage up to the death of the tree.
3. In this study we are interested in the effect of short-term freeze on one year old citrus.
 - a. In a short-term freeze, young citrus is more likely to die because it lacks the protection of a canopy.
 - b. Several technologies have been applied such as sprinkler systems to warm the tree.
 - c. A new technology is a tree enclosure that improves the effectiveness of the sprinkler.
4. The purpose of this study is to examine the economic benefits from the tree covers for young citrus.

B. Methodology

1. This study applies the basic results of the expected utility hypothesis to derive the insurance value of the tree enclosures.
2. Specifically, the study assumed two utility functions (the negative exponential and the power utility function) and used those functions to generate risk premiums.
 - a. Under the negative exponential:

$$U(W) = -\exp[-\delta W]$$

$$R_A(W) = -\frac{U''(W)}{U'(W)} = \delta$$

$$\begin{aligned} U^* &= -\exp(-\delta W^*) \\ \ln(-U^*) &= -\delta W^* \end{aligned} \tag{7}$$

$$W^* = -\frac{1}{\delta} \ln[-U^*]$$

- b. Similarly for the power utility function

$$\begin{aligned} U(W) &= \frac{W^{1-r}}{1-r} \\ R_R(W) &= r \\ U^* &= \frac{W^{*1-r}}{1-r} \end{aligned} \tag{8}$$

$$\begin{aligned} (1-r)U^* &= W^{*1-r} \\ W^* &= [(1-r)U^*]^{\frac{1}{1-r}} \end{aligned}$$

4. Now let us back up. I assume that the farmer possesses some utility function $Q(W)$ which we can approximate locally with either a negative exponential or a power utility function

$$\begin{aligned} R_A(W) &= -\frac{Q''(W)}{Q'(W)} = -q \\ R_R(W) &= -\frac{WQ''(W)}{Q'(W)} = -wq \end{aligned} \tag{9}$$

If the wealth of the individual is "large" or in the case where there is no debt both utility functions yield the same risk premium. However, if we assume that the farmer is leveraged W falls and the solution is different. Theoretically, the value of the insurance (cap) should be greater for more debt. Why?

C. Payoffs

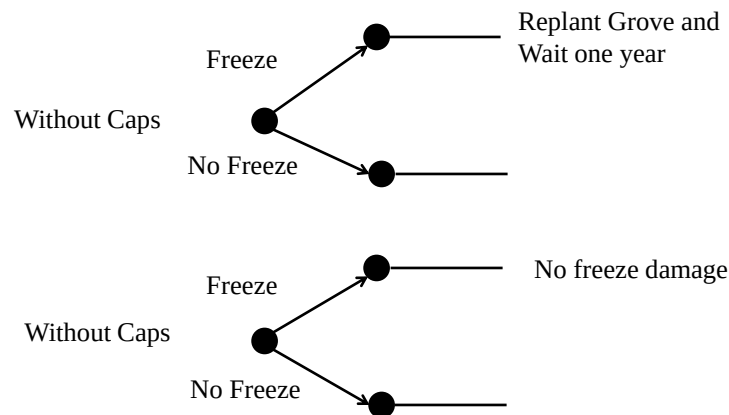


Figure 1: Orange Tree Caps

1. In this scenario, going without caps is a risky investment. With some probability the grower will experience a killing freeze in the first year that will for a replant and put off harvest one year.

$$E[U|a] = -\exp(-\delta W_F(a)) P[F] - \exp(-\delta W_{NF}(a)) (1 - P[F])$$

$$E[V|a] = \frac{W_F(a)^{1-r}}{1-r} P[F] + \frac{W_{NF}(a)^{1-r}}{1-r} (1 - P[F]) \quad (10)$$

where a is an action.

- a. With caps $W_F(a) = W_{NF}(a)$.
 - b. Without caps $W_F(a) < W_{NF}(a)$.
2. Thus,

$$MB_A(\delta) = -\frac{1}{\delta} \ln(E[U|cap]) - \left(-\frac{1}{\delta} \ln(E[U|nocap]) \right) \quad (11)$$

$$MB_R(r) = [(1-r)E[V|cap]]^{\frac{1}{1-r}} - [(1-r)E[V|nocap]]^{\frac{1}{1-r}}$$

3. The payoff for caps is the annualized profit for 25 years of orange production. The payoff without caps is the annualized profit of replanting the trees in the second year, delaying production one year and then producing orange for 25 years.
4. The total size of the production was fixed at 150 acres of orange. Thus, the different debt levels (0% , 25% , 50% , and 75 %) changes the equity of the producer. Total capital was \$ 1,312,500.

B. Results

III. Stochastic Dominance

- A. As previously mentioned, some have difficulties with the axioms underlying the expected utility hypothesis. These objections along with the objections of individuals who believe that it is impossible to know an individual's utility function has led to the development of risk efficiency criteria.
- B. One of the most general risk efficiency criteria is stochastic dominance.

1. The basic concept behind stochastic dominance is the comparison of two distribution functions with only very general rules about utility.
2. For example: First degree stochastic dominance assumes that more is preferred to less. Thus, the distribution F dominates the distribution G if

$$\begin{aligned} F(W) &\leq G(W) \text{ for all } W \text{ and} \\ F(W) &< G(W) \end{aligned} \quad (12)$$

where

$$\begin{aligned} F(W) &= \int_{-\infty}^W f(s) ds \text{ and} \\ G(W) &= \int_{-\infty}^W g(w) ds \end{aligned} \quad (13)$$

3. Intuitively, this implies that the probability of obtaining a wealth less than W is always lower for distribution F than for distribution G .
- C. Second Degree Stochastic Dominance also assumes that the decision maker is risk averse

$$\int_{\infty}^W [F(s) - G(s)] ds \geq 0 \quad (14)$$

for all W with $G(W) \neq F(W)$ for some W .

IV. Using Mean-Variance as a Choice Criteria

- A. If both alternatives are normally distributed another efficiency criteria is the mean-variance criteria.
- B. Assuming that the return is normally distributed with a mean of E and a variance σ^2 , the utility function can be written

$$U = U(R_j; E, \sigma). \quad (15)$$

1. Expected utility becomes

$$E(U) = \int_{-\infty}^{\infty} U(R) f(R; E, \sigma) dR \quad (16)$$

- a. We would like to show that the indifference curves of a risk averse investor can be expressed as a function of the mean and the standard deviation of returns.
- b. The indifference curve is the mapping of all combinations of risk and expected returns that yield the same utility level.
- c. Basically, we want to show that the indifference curves are increasing and convex.

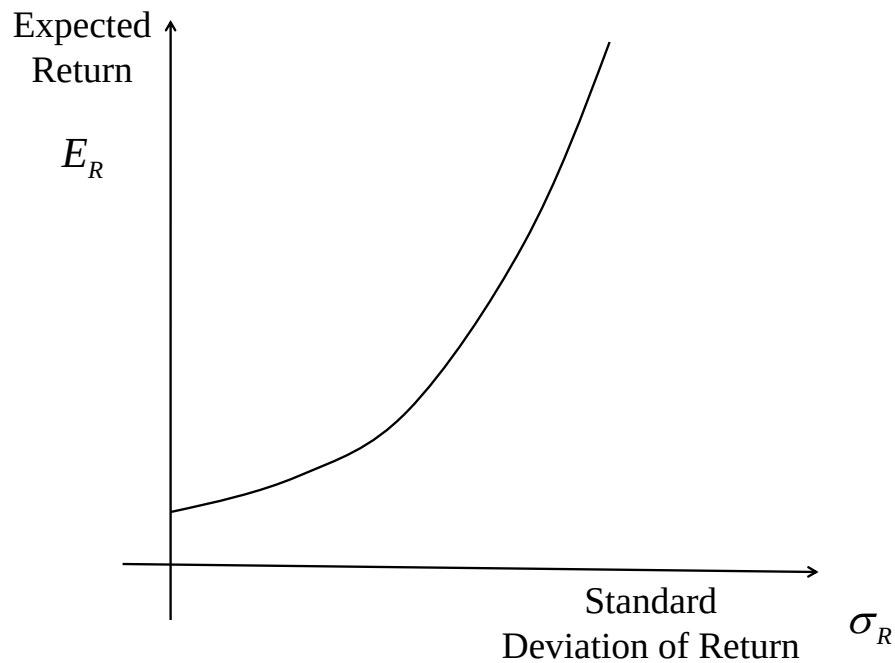


Figure 2: Utility under Normality

2. The first step in the proof is to transform the variable to a standard normal distribution

$$\tilde{z} = \frac{\tilde{R} - E}{\sigma} \Rightarrow \begin{cases} \tilde{R} \sim N(E, \sigma) \\ \tilde{z} \sim N(0, 1) \end{cases} \quad (17)$$

- a. Thus,

$$\tilde{R} = E + \sigma \tilde{z}, \quad \frac{d\tilde{R}}{d\tilde{z}} = \sigma, \quad d\tilde{R} = \sigma d\tilde{z}. \quad (18)$$

b. Hence, using a change in variables technique

$$E(U) = \int_{-\infty}^{\infty} U(E + \sigma \tilde{z}) f(z; 0, 1) dz \quad (19)$$

since $R = -\infty \Rightarrow z = -\infty$ and $R = \infty \Rightarrow z = \infty$.

c. Taking the derivative with respect to σ yields

$$\begin{aligned} \frac{dE(U)}{d\sigma} &= \int_{-\infty}^{\infty} U'(E + \sigma \tilde{z}) \left(\frac{dE}{d\sigma} + \tilde{z} \right) f(z; 0, 1) dz \\ &= \int_{-\infty}^{\infty} U'(E + \sigma \tilde{z}) \frac{dE}{d\sigma} f(z; 0, 1) dz + \int_{-\infty}^{\infty} U'(E + \sigma \tilde{z}) \tilde{z} f(z; 0, 1) dz \\ &= \frac{dE}{d\sigma} \int_{-\infty}^{\infty} U'(E + \sigma \tilde{z}) f(z; 0, 1) dz + \int_{-\infty}^{\infty} U'(E + \sigma \tilde{z}) \tilde{z} f(z; 0, 1) dz \end{aligned} \quad (20)$$

d. Setting $dE(U)/d\sigma = 0$ to map out an indifference curve

$$\frac{dE}{d\sigma} = - \frac{\int_{-\infty}^{\infty} U'(E + \sigma \tilde{z}) \tilde{z} f(z; 0, 1) dz}{\int_{-\infty}^{\infty} U'(E + \sigma \tilde{z}) f(z; 0, 1) dz} > 0 \quad (21)$$

- $U'(\cdot), f(\cdot) > 0$ – Thus, the denominator is always positive.
- The numerator is only positive if the utility function is convex. The sign depends on

$$- \int_{-\infty}^{\infty} U'(E + \sigma \tilde{z}) \tilde{z} f(z; 0, 1) dz. \quad (22)$$

Look at the pair $\pm z$ or $z^* = -z$ $f(z) = f(z^*)$, or the distribution is symmetric about zero. $U'(-z) > U'(z)$ as depicted in Figure 3. Thus, $-zU'(-z) < zU'(z)$ for every $-z < 0$. Thus,

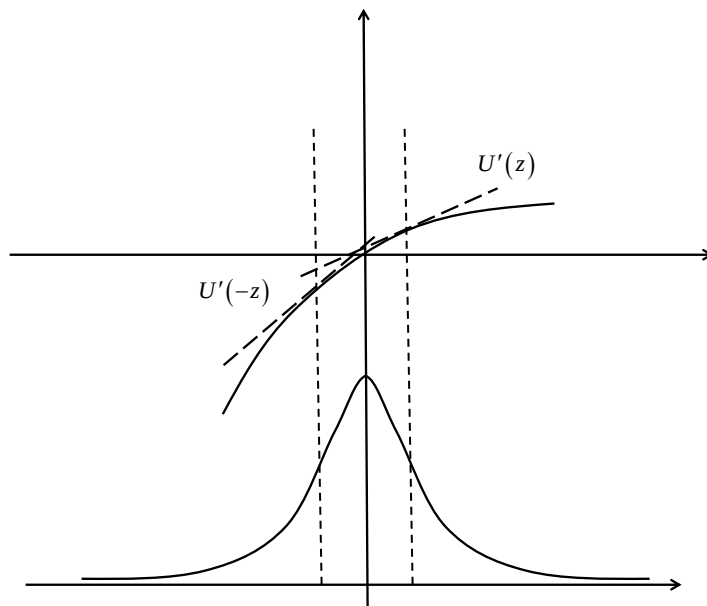


Figure 3: Convexity of the Utility Function

$$-\int_{-\infty}^{\infty} U'(\cdot) \tilde{z} f(\cdot) dz > 0 \quad (23)$$