

Lecture II: More Expected Utility, Risk Aversion, Absolute versus Relative Risk Aversion, and Arrow-Pratt

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I. Proof of the Expected Utility Hypothesis

- A. After some thought I have decided to present an overview of the proof of the expected utility hypothesis as found in Fama and Miller.
- B. To show that the expected utility rule follows from the axioms we must show that these axioms imply two things:
 - 1. There exists an order preserving utility function; that is, there exists a $U(.)$ such that $U(x) \geq U(y)$ implies $x \succsim y$ and $U(x) = U(y)$ implies $x \approx y$.
 - 2. The ordering of random prospects given by the function is according to the expected utility; that is $U(L = \{x, y\}, \alpha) = \alpha U(x) + (1 - \alpha) U(y)$.

C. Proof:

- 1. Assume that the space of all consumption bundles S is bounded by two prospects a and b such that $a \succ b$

$$\begin{aligned} a \succ x \succ b, \text{ or} \\ a \succsim x \succsim b \end{aligned} \tag{1}$$

- 2. Axioms 5 and 6 can be used to rank all the prospects in S in terms of the extreme prospects a and b .

- a. Specifically, using Axiom 5 we can define a probability $\alpha(x)$ such that

$$x \approx \{ (a, b), \alpha(x) \} \quad (2)$$

In this case $\alpha(x)$ is the probability of a gamble which makes the decision maker indifferent between the risky prospect of a and b and the certain bundle x .

- b. By similar definition $\alpha(y)$ is defined by $y \approx \{ (a, b), \alpha(y) \}$. By application of Axiom 6, $\alpha(x)$ is an order preserving utility function. Specifically,

$$\alpha(x) > \alpha(y) \Rightarrow x \succ y \quad (3)$$

[The risky prospects are the same (a, b) only the probabilities change.]

3. Thus, to show that the expected utility rule follows from the axioms, it remains only to show that $\alpha(x)$ ranks the random prospects according to expected utility.
- a. Consider a random prospect $\{ (x, y), \beta \}$ from Axiom 5 we know that there exists a unique probability $\alpha(\{ (x, y), \beta \})$ such that

$$\{ (x, y), \beta \} \approx \{ (a, b), \alpha(\{ (x, y), \beta \}) \} \quad (4)$$

That is, there exists a probability $\alpha(\{ (x, y), \beta \})$ that leaves the individual indifferent between the gamble $\{ (x, y), \beta \}$ and the gamble $\{ (a, b), \alpha(\{ (x, y), \beta \}) \}$.

- b. We already know that $\alpha(\{ (x, y), \beta \})$ ranks $\{ (x, y), \beta \}$ relative to other prospects in S . It remains to be shown that

$$\alpha(\{ (x, y), \beta \}) = \beta\alpha(x) + (1 - \beta)\alpha(y) \quad (5)$$

- c. Because $x \approx \{ (a, b), \alpha(x) \}$ from Axiom 4, we can conclude that by independence

$$\{ (x, y), \beta \} \approx \{ \{ (a, b), \alpha(x) \}, y, \beta \} \quad (6)$$

Similarly, by Axiom 4

$$\begin{aligned} \{ (\{ (a, b), \alpha(x) \}, y), \beta \} &\approx \\ \{ (\{ (a, b), \alpha(x) \}, \{ (a, b), \alpha(y) \}), \beta \} \end{aligned} \quad (7)$$

Finally, applying Axiom 3 (Transitivity)

$$\{ (x, y), \beta \} \approx \{ (\{ (a, b), \alpha(x) \}, \{ (a, b), \alpha(y) \}), \beta \} \quad (8)$$

- d. Think about the payoffs of this gamble. What is the probability of getting a ? a is obtained with probability $\alpha(x)$ in the first sub-gamble and $\alpha(y)$ in the second sub-gamble. In addition, the probability of winning the first gamble is β and the second gamble is $1 - \beta$. Thus,

$$P[a] = \alpha(x)\beta + \alpha(y)(1 - \beta). \quad (9)$$

Similarly,

$$P[b] = (1 - \alpha(x))\beta + (1 - \alpha(y))(1 - \beta) \quad (10)$$

Check $P[a] + P[b] = 1$.

- e. Thus, the composite gamble is identical to

$$\{ (a, b), \beta\alpha(x) + (1 - \beta)\alpha(y) \} \quad (11)$$

Hence,

$$\{ (x, y), \beta \} \approx \{ (a, b), \beta\alpha(x) + (1 - \beta)\alpha(y) \}. \quad (12)$$

However, $\{ (x, y), \beta \} \approx \{ (a, b), \alpha(\{ (a, b), \beta \}) \}$. Thus, by Axiom 5 $\alpha(\{ (a, b), \beta \}) = \beta\alpha(x) + (1 - \beta)\alpha(y) - \alpha(\cdot)$ is ranked according to the expected utility.

II. Establishing a Definition of Risk Aversion

- A. A useful way to begin a discussion of risk aversion is to compare three simply utility functions that assume more wealth or income is preferred to less. Each of the possibilities are presented in Figure 1 where the gambles were previously referred to as $\{ (a, b), \alpha \}$.

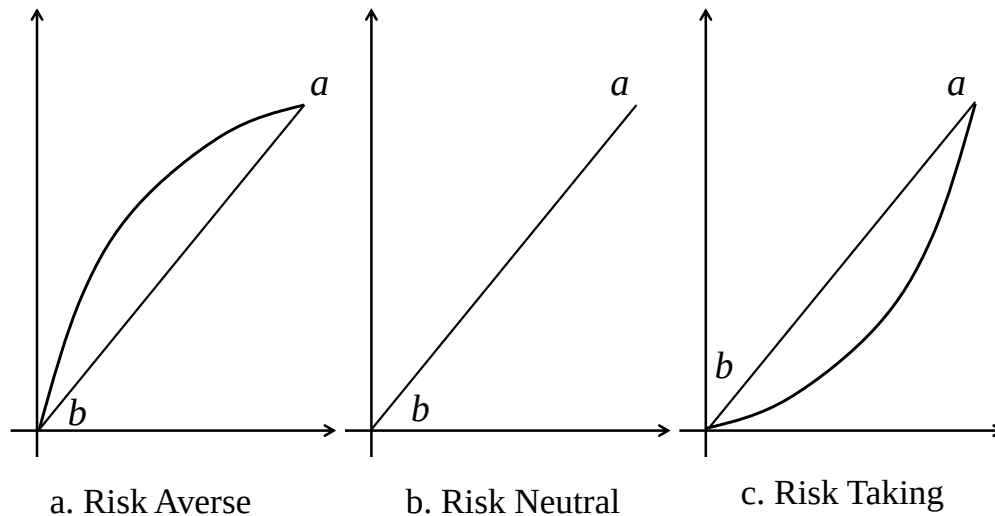


Figure 1: Risk Attitudes

1. The question now becomes whether the individual will prefer the actuarial value of the gamble or the gamble itself.
2. The utility of the gamble is

$$E[U(w)] = \alpha U(a) + (1 - \alpha) U(b) \quad (13)$$

which is on the arc between the points a and b on the utility function.

3. The expected or actuarial value of the gamble is

$$E[w] = \alpha a + (1 - \alpha) b. \quad (14)$$

The utility of the actuarial value is then

$$U(\alpha a + (1 - \alpha) b). \quad (15)$$

4. Graphically, the expected utility and the utility of the actuarial payoff lie on a vertical line from $\bar{w} = \alpha a + (1 - \alpha) b$ as depicted in Figure 2.

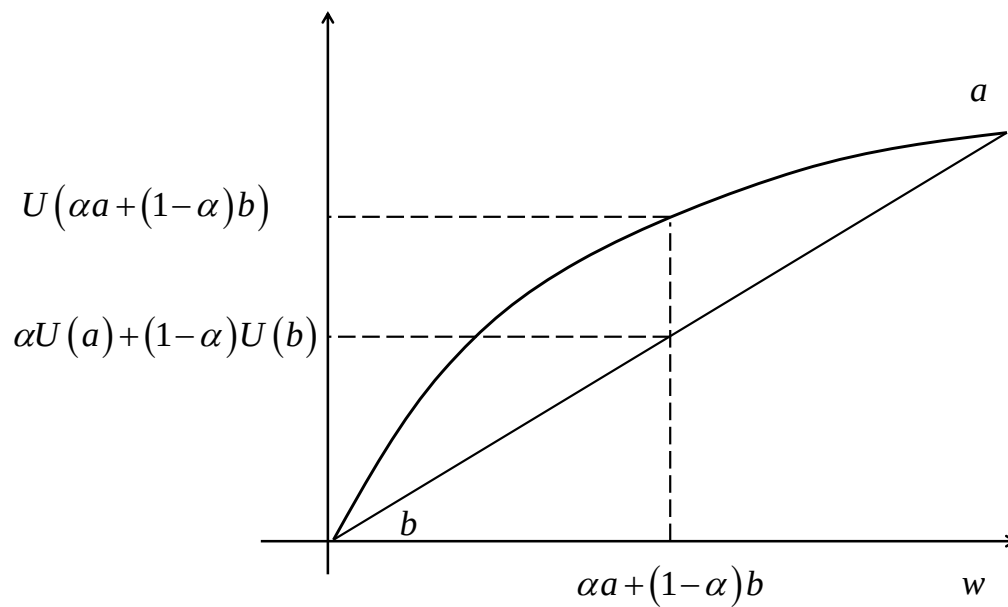


Figure 2: Expected Utility and Actuarial Value

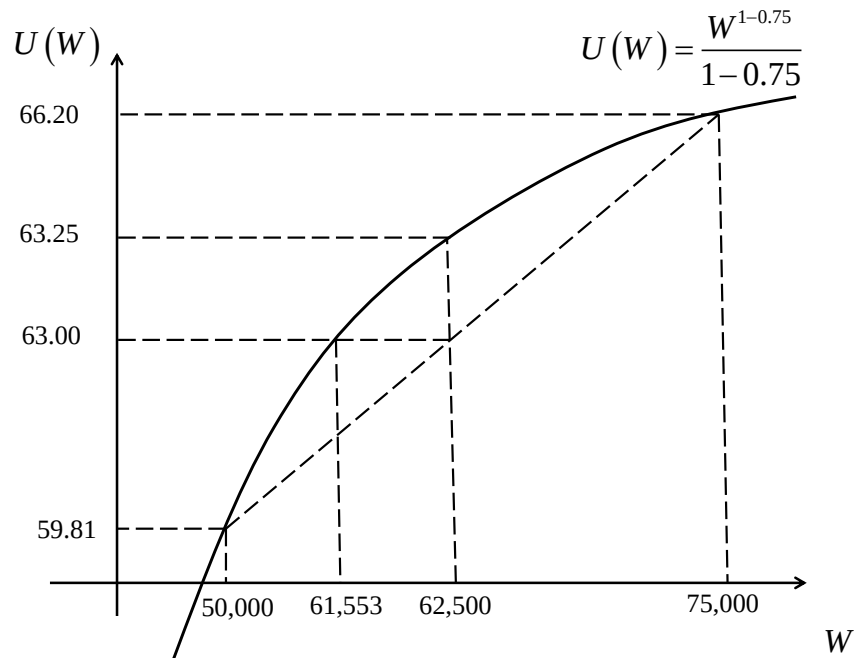
B. Risk Aversion

1. In the risk averse case

$$U[E(w)] > E[U(w)] \quad \text{or} \quad U(\alpha a + (1 - \alpha) b) > \alpha U(a) + (1 - \alpha) U(b) \quad (16)$$

Thus, the individual derives less utility from the gamble than the actuarial payoff.

2. In the risk neutral case, the expected utility and the utility from the actuarial payoff are equal



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Figure 3: Graphical Overview of Expected Utility

$$U[E(w)] = E[U(w)] \text{ or } U(\alpha a + (1 - \alpha)b) = \alpha U(a) + (1 - \alpha)U(b) \quad (17)$$

3. In the risk loving case, the expected utility is greater than the utility of the actuarial payoff

$$U[E(w)] < E[U(w)] \text{ or } U(\alpha a + (1 - \alpha)b) < \alpha U(a) + (1 - \alpha)U(b) \quad (18)$$

- C. Figure 3 presents a graphical overview of the expected utility problem.

* First we define the certainty equivalent as the certain level of income that will yield the same utility as the risky gamble. Using the power utility function

$$\begin{aligned} U &= \frac{W^{1-r}}{1-r} \\ U(1-r) &= W^{1-r} \\ [u(1-r)]^{\frac{1}{1-r}} &= W^* \end{aligned} \quad (19)$$

where r is the risk aversion coefficient, U is the expected utility generated by the gamble, and W^* is the certainty equivalent.

- * For example, consider the gamble paying \$ 50,000 and \$ 75,000 – each with a probability of 0.5. The expected utility of the gamble given a risk aversion coefficient of 0.75 can be computed as

$$\begin{aligned} E[U(W)] &= 0.5 \left[\frac{50,000^{1.0-0.75}}{1.0-0.75} \right] + 0.5 \left[\frac{75,000^{1.0-0.75}}{1.0-0.75} \right] \\ &= 0.5(59.81) + 0.5(66.20) = 63.00 \end{aligned} \quad (20)$$

The certainty equivalent can then be computed as

$$W^* = [(1.0 - 0.75) 63.00]^{\frac{1}{1.0-0.75}} = 61,553 \quad (21)$$

- * Notice that the certainty equivalent is less than the expected value of the gamble

$$E[W] = 0.5(50,000) + 0.5(75,000) = 62,500. \quad (22)$$

- D. The difference between the expected value of the gamble and the certainty equivalent is defined as the risk premium or Markowitz risk premium

$$\pi = E[W] - W^* = 62,500 - 61,552 = 947. \quad (23)$$

The other convention which is called the cost of the gamble is the difference between the current wealth and the certainty equivalence of the gamble.

III. Absolute and Relative Risk Aversion

- A. In talking about the absolute and relative risk aversion, it is useful to discuss the Arrow-Pratt risk aversion coefficients and risk aversion in the small and large.
- B. First, we need to develop an alternative measure of risk premium.
 1. Let us begin by examining a bet with expected value $\bar{z}$ and variance σ_z^2 . The total wealth of the individual after the bet is $W + z$. The wealth after the bet is distributed $N(w + \bar{z}, \sigma_z^2)$.
 2. We seek to determine the risk premium $\pi(W, z)$ that leaves the individual indifferent between the uncertain gamble and $W + \bar{z} - \pi(W, z)$ – a certain prospect

$$E[U(W + z)] = U[W + \bar{z} - \pi(W, z)]. \quad (24)$$

To simplify matters assume that W and $\bar{z}$ are both zero. Thus, π is the amount of money that must be taken away from the expected value to leave the decision maker unaffected by risk

$$U(W - \pi) = E[U(W + z)] \quad (25)$$

Taking the Taylor series expansion of each side

$$\begin{aligned} U(W - \pi) &= U(W) + U'(W)[W - \pi] + R_1 \\ &\Rightarrow U(W) - \pi U'(W) \\ E[U(W + z)] &= E\left[U(W) + U'(W)[W + z] + \frac{1}{2}U''(W)[W + z]^2 + R_2\right] \\ &\Rightarrow U(W) + \frac{1}{2}U''(W)\sigma_z^2 \\ \therefore U(W) - \pi U'(W) &= U(W) + \frac{1}{2}U''(W)\sigma_z^2 \\ &\Rightarrow \pi = -\frac{1}{2}\sigma_z^2 \left[\frac{U''(W)}{U'(W)} \right] \end{aligned} \quad (26)$$

3. The last term is called the Arrow-Pratt Absolute Risk aversion coefficient

$$R_A = -\frac{U''(W)}{U'(W)} \quad (27)$$

An alternative measure is referred to as the Arrow-Pratt Relative Risk aversion coefficient

$$R_R = -\frac{WU''(W)}{U'(W)} \quad (28)$$

C. Risk aversion in the large and small

1. The basic concept is that the Arrow-Pratt approach is valid for symmetric distributions of small bets. As the distribution skews, the Markowitz risk premium is always valid while the Arrow-Pratt worsens.
2. Assume an investor with wealth of 100,000 enters into a bet that loses 1 % with probability 0.5 and gains 1.5 % with probability 0.5.
 - Using the Markowitz approach assuming $U(W) = \ln(W)$

$$\begin{aligned} E[U(W)] &= 0.5 \times \ln(99,000) + 0.5 \times \ln(101,500) \\ &= 0.5 \times [11.503 + 11.528] = 11.515 \end{aligned}$$

$$\exp(11.515) = 100,242.20$$

$$E[W] = 100,250.00$$

$$\pi_M = 100,250.00 - 100,242.20 = 7.80$$

(29)

- Using Arrow-Pratt

$$\begin{aligned} \mu_z &= -1,000 \times 0.5 + 1,500 \times 0.5 = 250 \\ \sigma_z^2 &= 0.5 \times (-1,000 - 250)^2 + 0.5 \times (1,500 - 250)^2 = 1,562,500 \end{aligned}$$

$$\left. \begin{aligned} U(W) &= \ln(W) \\ U'(W) &= \frac{1}{W} \\ U''(W) &= -\frac{1}{W^2} \end{aligned} \right\} \Rightarrow -\frac{U''(W)}{U'(W)} = \frac{1}{W}$$

$$\therefore \pi_P = \frac{1}{2} (1,562,500) \left(\frac{1}{100,000} \right) = 7.81$$

(30)

3. Repeat this exercise, but assume the gamble is $\{ (-25,000, 30,000), 0.7 \}$ with an initial wealth of \$ 100,000 – $\pi_m = 3,044.29$ and $\pi_p = 3,176.25$. The difference is 131.96.