

Lecture I: Linkage Between Utility in Finance and Utility in Economics and Axioms of Expected Utility

Charles B. Moss

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I. Utility in Finance and Economics

A. From your first course in economics, you were exposed to the idea of the utility function. Some may have heard the terms cardinal versus ordinal utility. Historically, the cardinal or absolute utility measure is older, but requires more stringent assumptions. Ordinal, on the other hand is sufficient for most supply and demand derivations.

1. Axioms of Ordinal Utility

- **Axiom 1** (*Completeness*): For every pair of vectors $x \in R^+$ and $z \in R^+$ either $x \succsim z$ or $z \succsim x$.
- **Axiom 2** (*Reflexivity*): For every $x \in R^+$ $x \succsim x$.
- **Axiom 3** (*Transitivity*): If $x \succsim y$ and $y \succsim z$, then $x \succsim z$.
- **Axiom 4** (*Continuity*): For every $x \in R^+$ the two subsets of all strictly preferred and all strictly worse bundles are both open.

2. Given these four axioms, there exists an ordinal utility function which describes the individual preferences

$$\begin{aligned} U(x) > U(z) &\Leftrightarrow x \succ z \\ U(x) = U(z) &\Leftrightarrow x \approx z \end{aligned} \tag{1}$$

3. Coupled with the assumption that $U(x)$ is twice differentiable and strictly concave, these axioms are typically sufficient for

the derivation of the consumer's problem. In addition, the emphasis on ordinal functions leaves the optimal solution invariant to positive monotonic transformation.

- B. The basic consumer problem is to maximize utility subject to a budget constraint. The problem also has a mathematical dual which is to minimize expenditures to obtain a given level of utility. This duality is of frequent use in financial economics. Hence, we will spend some time developing the concept of duality in consumption. In addition, the indirect utility function can be viewed as the forerunner of the cardinal utility of wealth which will be developed later.

1. As previously stated, the goal of the consumer is to maximize utility subject to a budget constraint

$$\begin{aligned} \text{Max}_{x_1, x_2} U(x_1, x_2) &= x_1^\alpha x_2^\beta \\ \text{s.t. } P_1 x_1 + P_2 x_2 &\leq Y \end{aligned} \quad (2)$$

- a. Mathematically one can solve for the Marshallian demand curves using the Lagrange multiplier

$$\begin{aligned} L &= x_1^\alpha x_2^\beta + \lambda (Y - P_1 x_1 - P_2 x_2) \\ L_1 &= \alpha \frac{x_1^{\alpha-1} x_2^\beta}{x_1} - \lambda P_1 = 0 \\ L_2 &= \beta \frac{x_1^\alpha x_2^{\beta-1}}{x_2} - \lambda P_2 = 0 \\ L_\lambda &= Y - P_1 x_1 - P_2 x_2 = 0 \end{aligned} \quad (3)$$

Using L_1 and L_2

$$\begin{aligned} \frac{\alpha \frac{x_1^{\alpha-1} x_2^\beta}{x_1}}{\beta \frac{x_1^\alpha x_2^{\beta-1}}{x_2}} &= \frac{\lambda P_1}{\lambda P_2} \\ \Rightarrow \frac{\alpha x_2}{\beta x_1} &= \frac{\lambda P_1}{\lambda P_2} \\ \Rightarrow x_2 &= \frac{\beta P_1}{\alpha P_2} x_1 \end{aligned} \quad (4)$$

Using L_λ

$$\begin{aligned}
 P_1 x_1 + P_2 \left[\frac{\beta P_1}{\alpha P_2} x_1 \right] &= Y \\
 \left[1 + \frac{\beta}{\alpha} \right] P_1 x_1 &= Y \\
 x_1^*(P_1, P_2, Y) &= \frac{\alpha Y}{[\alpha + \beta] P_1} \\
 x_2^*(P_1, P_2, Y) &= \frac{\beta P_1}{\alpha P_2} \left[\frac{\alpha Y}{[\alpha + \beta] P_1} \right] = \frac{\beta Y}{[\alpha + \beta] P_2}
 \end{aligned} \tag{5}$$

These are the Marshallian or income constant demand curves.

- b. Using the Marshallian demand curves and the original specification of the utility function, it is possible to derive the indirect utility function that indicates the maximum level of utility that can be derived from a given income level and vector of prices.

$$\begin{aligned}
 V(P_1, P_2, Y) &= x_1^*(P_1, P_2, Y)^\alpha x_2^*(P_1, P_2, Y)^\beta \\
 &= \left[\frac{\alpha Y}{[\alpha + \beta] P_1} \right]^\alpha \left[\frac{\beta Y}{[\alpha + \beta] P_2} \right]^\beta \\
 &= \left(\frac{\alpha}{P_1} \right)^\alpha \left(\frac{\beta}{P_2} \right)^\beta \left(\frac{Y}{\alpha + \beta} \right)^{\alpha + \beta}
 \end{aligned} \tag{6}$$

Notice that we could rewrite this as

$$\begin{aligned}
 U(Y) &= k Y^{\alpha + \beta} \\
 \text{where } k &= \left(\frac{\alpha}{P_1} \right)^\alpha \left(\frac{\beta}{P_2} \right)^\beta \frac{1}{[\alpha + \beta]^{\alpha + \beta}}
 \end{aligned} \tag{7}$$

2. To complete the dual specification we consider the flip side of the consumer's problem – namely the minimum level of expenditures required to obtain a given level of utility.
- a. Mathematically

$$\begin{aligned} \text{Min}_{x_1, x_2} P_1 x_1 + P_2 x_2 \\ \text{s.t. } U^* = x_1^\alpha x_2^\beta \end{aligned} \quad (8)$$

Again, building the Lagrange multiplier formulation

$$\begin{aligned} L &= P_1 x_1 + P_2 x_2 + \mu (U^* - x_1^\alpha x_2^\beta) \\ L_1 &= P_1 - \mu \alpha \frac{x_1^{\alpha-1} x_2^\beta}{x_1} = 0 \\ L_2 &= P_2 - \mu \beta \frac{x_1^\alpha x_2^{\beta-1}}{x_2} = 0 \\ L_\mu &= U^* - x_1^\alpha x_2^\beta = 0 \end{aligned} \quad (9)$$

Using L_1 and L_2

$$\frac{P_1}{P_2} = \frac{\mu \alpha \frac{x_1^{\alpha-1} x_2^\beta}{x_1}}{\mu \beta \frac{x_1^\alpha x_2^{\beta-1}}{x_2}} = \frac{\alpha x_2}{\beta x_1} \Rightarrow x_1 = \frac{\alpha}{\beta} \frac{P_2}{P_1} x_2 \quad (10)$$

Using L_μ

$$\begin{aligned} U^* - \left(\frac{\alpha P_2}{\beta P_1} x_2 \right)^\alpha x_2^\beta &= 0 \\ U^* - \left(\frac{\alpha P_2}{\beta P_1} \right) x_2^{\alpha+\beta} &= 0 \\ x_2^{\alpha+\beta} &= \left(\frac{\beta P_1}{\alpha P_2} \right)^\alpha U^* \end{aligned} \quad (11)$$

$$x_2^h(P_1, P_2, U) = \left(\frac{\beta P_1}{\alpha P_2} \right)^{\frac{\alpha}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}}$$

Solving for the demand for good 1

$$\begin{aligned}
 x_1^h(P_1, P_2, U^*) &= \left(\frac{\alpha P_2}{\beta P_1} \right) \left(\frac{\beta P_1}{\alpha P_2} \right)^{\frac{\alpha}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}} \\
 &= \left(\frac{\alpha P_2}{\beta P_1} \right)^{\frac{\alpha+\beta}{\alpha+\beta}} \left(\frac{\beta P_1}{\alpha P_2} \right)^{\frac{\alpha}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}} \\
 &= \left(\frac{\alpha P_2}{\beta P_1} \right)^{\frac{\alpha+\beta}{\alpha+\beta}} \left(\frac{\alpha P_2}{\beta P_1} \right)^{\frac{-\alpha}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}} \\
 &= \left(\frac{\alpha P_2}{\beta P_1} \right)^{\frac{\beta}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}}
 \end{aligned} \tag{12}$$

These are the Hicksian or utility constant demand curves.

- c. Finally, the last piece to link the primal and the dual is the expenditure function

$$\begin{aligned}
 e(P_1, P_2, U) &= P_1 \left[\frac{\alpha P_2}{\beta P_1} \right]^{\frac{\beta}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}} + P_2 \left[\frac{\beta P_1}{\alpha P_2} \right]^{\frac{\alpha}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}} \\
 &= \left[P_1^{\frac{\alpha}{\alpha+\beta}} \left[\frac{\alpha P_2}{\beta} \right]^{\frac{\beta}{\alpha+\beta}} + P_2^{\frac{\beta}{\alpha+\beta}} \left[\frac{\beta P_1}{\alpha} \right]^{\frac{\alpha}{\alpha+\beta}} \right] U^{*\frac{1}{\alpha+\beta}} \\
 &= \left(\left[\frac{\alpha}{\beta} \right]^{\frac{\beta}{\alpha+\beta}} + \left[\frac{\beta}{\alpha} \right]^{\frac{\alpha}{\alpha+\beta}} \right) P_1^{\frac{\alpha}{\alpha+\beta}} P_2^{\frac{\beta}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}}
 \end{aligned} \tag{13}$$

Notice that

$$\begin{aligned}
 \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} &= \alpha^{-\frac{\alpha}{\alpha+\beta}} \alpha^{\frac{\alpha}{\alpha+\beta}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \right] \\
 &= \alpha^{-\frac{\alpha}{\alpha+\beta}} \left[\frac{\alpha^{\frac{\alpha+\beta}{\alpha+\beta}}}{\beta^{\frac{\beta}{\alpha+\beta}}} + \beta^{\frac{\alpha}{\alpha+\beta}} \right] \\
 &= \alpha^{-\frac{\alpha}{\alpha+\beta}} \beta^{-\frac{\beta}{\alpha+\beta}} [\alpha + \beta] \\
 \therefore e(P_1, P_2, U) &= \left(\frac{P_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{P_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} U^{*\frac{1}{\alpha+\beta}} [\alpha + \beta]
 \end{aligned} \tag{14}$$

3. The dual specification becomes apparent if $U^* = V$ and $e = Y$. From the above specification

$$\begin{aligned}
 Y &= \left(\frac{P_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{P_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} V^{\frac{1}{\alpha+\beta}} [\alpha + \beta] \\
 \left(\frac{\alpha}{P_1}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{\beta}{P_2}\right)^{\frac{\beta}{\alpha+\beta}} \frac{Y}{\alpha + \beta} &= V^{\frac{1}{\alpha+\beta}} \\
 \left(\frac{\alpha}{P_1}\right)^{\alpha} \left(\frac{\beta}{P_2}\right)^{\beta} \left[\frac{Y}{\alpha + \beta}\right]^{\alpha+\beta} &= V
 \end{aligned} \tag{15}$$

Thus, one of these expressions can be derived from the other.

II. Axioms of Expected Utility

- A. As stated in the proceeding section, the four axioms are sufficient for establishing ordinal results. However, the bulk of expected utility theory rests on cardinal results where we want to compare different utility levels. In addition, we want to change our terminology from bundles (or complexes) to lotteries.
- B. A lottery is defined as an action that has alternative outcomes.
 1. Copeland and Weston – $G(x, z; \alpha)$, the decision maker gets x with probability α and z with probability $1 - \alpha$.

2. Ingersoll – $L_1 = \{ (x_1, x_2, \dots, x_m), \pi \}$ where x is a vector of goods and $\pi = (\pi_1, \pi_2, \dots, \pi_m)$ is a vector of probabilities.

C. Axioms for Expected Utility

- * **Axiom 1** (Completeness): For every pair of lotteries $L_1 \succsim L_2$ or $L_2 \succsim L_1$.
- * **Axiom 2** (Reflexivity): For every lottery $L_1 \succsim L_1$.
- * **Axiom 3** (Transitivity): If $L_1 \succsim L_2$ and $L_2 \succsim L_3$, then $L_1 \succsim L_3$.
- * **Axiom 4** (Independence): If $L_1 = \{ (x_1, x_2, \pi) \}$ and $L_2 = \{ (x_1, x_3), \pi \}$. If $x_1 \approx x_3$ then $L_1 \approx L_2$. This holds if x_3 is another gamble.
- * **Axiom 5** (Measurability or Continuity): If $x_1 > x_2 > x_3$ then there exists a probability π , $0 \leq \pi \leq 1$, such that $x_2 \approx \{ (x_1, x_2), \pi \}$.
- * **Axiom 6** (Ranking or Dominance): Let $L_1 = \{ (x_1, x_2), \pi_1 \}$ and $L_2 = \{ (x_1, x_2), \pi_2 \}$. If $x_1 > x_2$, then $L_1 \succsim L_2$ if and only if $\pi_1 > \pi_2$.

- D. Under these axioms the choice made by a decision maker faced with selecting between two or more lotteries will be the one with the highest expected utility. That is, the choice maximizes $\sum_i \pi_i U(x_i)$ where $U(x_i)$ is a particular cardinal utility function.