

Lecture XIX: Optimal Debt – The Risk Balancing Model

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Readings for next time

Agricultural Policy and the Risk-Balancing Model

Optimal Leverage and Capital Gains

Stochastic Optimal Control

Readings for Wednesday

- ▶ Ahrendsen, B.L., R.N. Collender, and B.L. Dixon. 1994. An Empirical Analysis of Optimal Farm Capital Structure Decisions. *Agricultural Finance Review* 54, 108-119.
- ▶ Jensen, F.E. and L.N. Langemeier. 1996. Optimal Leverage with Risk Aversion: Empirical Evidence. *Agricultural Finance Review* 56, 85-97.
- ▶ Moss, C.B., J.S. Shonkwiler, and S.A. Ford. 1990. A Risk Endogenous Model of Aggregate Agricultural Debt. *Agricultural Finance Review* 50, 73-79.
- ▶ Moss, C.B. and A.K. Mishra. 2013. Debt-Equity Structure and Risk Balancing by Farm Businesses. Food and Resource Economics Department, University of Florida, Working Paper.

Readings for Friday

- ▶ Darrough, M.N. and N.M. Stoughton. 1986. Moral Hazard and Adverse Selection: The Question of Financial Structure. *Journal of Finance* 41(2), 501-513.
- ▶ Innes, R.D. 2002. Market Failures and Second-Best Analysis with a Focus on Nutrition, Credit, and Incomplete Markets. *Handbook on Agricultural Economics*. Elsevier, pp. 1828-1892.
- ▶ Innes, R.D. 1990. Limited Liability and Incentive Contracting with Ex-Ante Action Choices. *Journal of Economic Theory* 52(1), 45-67.
- ▶ Stiglitz, J.E. and A. Weiss. 1981. Credit Rationing in Markets with Imperfect Information. *American Economic Review* 71(3), 393-410.

Agricultural Policy and the Risk-Balancing Model

- ▶ Featherstone, A.M., C.B. Moss, T.G. Baker, and P.V. Preckel. 1988. The Theoretical Effects of Farm Policies on Optimal Leverage and the Probability of Equity Loss. *American Journal of Agricultural Economics* 70(3), 572-579.
- ▶ The basic point of Featherstone et al. was that farm programs could lead to increased leverage by reducing the business risk.

- As a part of this development, remember the financial risk as formulated in Collins's model

$$\sigma_E^2 = \frac{\sigma_A^2}{(1 - \delta)^2} \quad (1)$$

and the optimal debt ratio

$$\delta^* = 1 - \left[\frac{\rho \sigma_A^2}{\mu_A - K} \right] \quad (2)$$

- Solving for $1 - \delta^*$ using Equation 2

$$\frac{1}{1 - \delta^*} = \frac{\mu_A - K}{\rho \sigma_A^2} \quad (3)$$

and substituting the result from Equation 3 into the variance for the rate of return on equity yields

$$\sigma_{E^*}^2 = \left[\frac{\mu_A - K}{\rho} \right]^2 \frac{1}{\sigma_A^2} \quad (4)$$

which is the variance of the rate of return on equity given that leverage is optimally chosen.

- ▶ Thus, the effect of changes in the business risk on the risk of the rate of return on equity can be derived as

$$\frac{\partial \sigma_{E^*}^2}{\partial \sigma_A^2} = - \left(\frac{\mu_A - K}{\rho \sigma_A^2} \right)^2 < 0. \quad (5)$$

- ▶ Returning to the rate of return on equity

$$\mu_E = \frac{\mu_A - K\delta}{1 - \delta} \quad (6)$$

$$\Rightarrow \frac{\mu_A - K + K - K\delta}{1 - \delta} = \frac{\mu_A - K}{1 - \delta} + K \frac{1 - \delta}{1 - \delta}.$$

Substituting for the optimal $1 - \delta^*$ from Equation 3

$$\mu_E = \frac{(\mu_A - K)^2}{\rho \sigma_A^2} + K. \quad (7)$$

- Taking the derivative of Equation 7

$$\frac{\partial \mu_E}{\partial \sigma_A^2} = -\frac{(\mu_A - K)^2}{\rho (\sigma_A^2)^2} < 0. \quad (8)$$

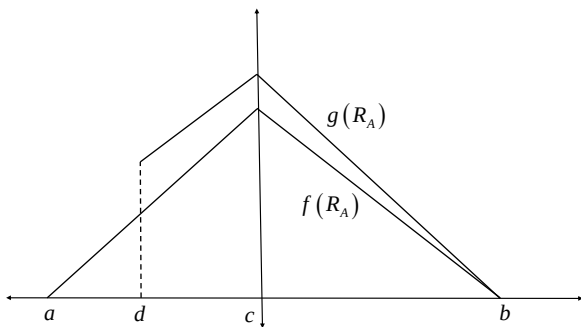
- ▶ Returning to the probability of equity loss – Featherstone et al. formulate the probability of equity loss as

$$\Pr(R_E < c_L) = \Pr\left(\frac{R_E - \mu_E}{\sigma_E} < \frac{c_L - \mu_E}{\sigma_E}\right) \quad (9)$$

- ▶ Rewriting this slightly

$$\Pr(R_E < c_L) = \Pr\left(\frac{R_E - \mu_E}{\sigma_E} < Z\right) \quad (10)$$
$$Z = \frac{(c_L - K) \rho \sigma_A^2 - (\mu_A - K)^2}{\sigma_A (\mu_A - K)}$$

Mean and Mode Preserving Spread for a Triangular Distribution



Optimal Leverage and Capital Gains

- ▶ Moss, C.B., S.A. Ford, and W.G. Boggess. 1989. Capital Gains, Optimal Leverage, and the Probability of Equity Loss: A Theoretical Model. *Agricultural Finance Review* 49, 1927-34.
- ▶ The basic concept in Moss et al. is whether the certain aspects of the tax code changes the optimal debt. Specifically, I was interested in the capital gains tax break.

- ▶ Adding income tax considerations to the rate of return to equity

$$R_E = \left[\frac{r_P (1 - \tau)}{A} + (1 - \psi\tau) i - K (1 - \tau) \delta \right] \frac{1}{1 - \delta} \quad (11)$$

- ▶ τ is the marginal tax rate.
- ▶ ψ is the capital gains exclusion.

- Taking the derivative with respect to δ

$$\frac{\partial V(\delta)}{\partial \delta} = -\frac{K(1-\tau)}{(1-\delta)} + \frac{[\mu_A - K(1-\tau)\delta]}{(1-\delta)^2} - \frac{\rho\sigma_A^2}{(1-\delta)^3} = 0. \quad (14)$$

After a little algebra, this expression becomes

$$\delta^* = 1 - \frac{\rho\sigma_A^2}{\mu_A - K(1-\tau)}. \quad (15)$$

- I probably should have rewritten Equation 15 as

$$\delta^* [\sigma_A^2(\tau, \psi), \mu(\tau, \psi)] = 1 - \frac{\rho \sigma_A^2(\tau, \psi)}{\mu_A(\tau, \psi) - K(1 - \tau)}. \quad (16)$$

- To examine the effect of tax policy on optimal leverage and the probability of equity loss, we can start by totally differentiating Equation 16

$$\begin{aligned}
 d\delta^* \left[\sigma_A^2(\tau, \psi), \mu_A(\tau, \psi) \right] &= - \frac{\rho}{[\mu_A(\tau, \psi) - K(1 - \tau)]} d\sigma_A^2(\tau, \psi) \\
 &+ \frac{\rho \sigma_A^2(\tau, \psi)}{[\mu_A(\tau, \psi) - K(1 - \tau)]^2} d\mu_A(\tau, \psi) + K \frac{\rho \sigma_A^2(\tau, \psi)}{[\mu_A(\tau, \psi) - K(1 - \tau)]^2} d\tau \quad (17) \\
 &\quad - \frac{\rho [1 - \tau] \sigma_A^2(\tau, \psi)}{[\mu_A(\tau, \psi) - K(1 - \tau)]^2} dK
 \end{aligned}$$

- Now for a couple of corrections – did this reduce the acceptability and impact of this article?

- First, let us set $d\tau = dK = 0$ or hold the tax rate (not the capital gains exclusion) and the interest rate constant. In this case Equation 17 becomes

$$d\delta^* \left[\sigma_A^2(\tau, \psi), \mu_A(\tau, \psi) \right] \Big|_{d\tau=dK=0} = - \frac{\rho}{[\mu_A(\tau, \psi) - K(1 - \tau)]} d\sigma_A^2(\tau, \psi) \\ + \frac{\rho \sigma_A^2(\tau, \psi)}{[\mu_A(\tau, \psi) - K(1 - \tau)]^2} d\mu_A(\tau, \psi) \quad (18)$$

- Given Equation 18 we can derive the change in leverage from a change in the capital gains exclusion (note the typo in Equation 10 in the original article)

$$\left. \frac{d\delta^* [\sigma_A^2(\tau, \psi), \mu_A(\tau, \psi)]}{d\psi} \right|_{d\tau=dK=0} = - \frac{\rho}{[\mu_A(\tau, \psi) - K(1 - \tau)]} \left. \frac{d\sigma_A^2(\tau, \psi)}{d\psi} \right|_{d\tau=0}$$

$$\frac{\rho\sigma_A^2(\tau, \psi)}{[\mu_A(\tau, \psi) - K(1 - \tau)]^2} \left. \frac{d\mu_A(\tau, \psi)}{d\psi} \right|_{d\tau=0} \geq 0$$

(19)

- ▶ Looking at each part

$$\begin{aligned} \left. \frac{\partial \mu_A(\tau, \psi)}{\partial \psi} \right|_{d\tau=0} &= -\tau \mu_i < 0 \\ \left. \frac{\partial \sigma_A^2(\tau, \psi)}{\partial \psi} \right|_{d\tau=0} &= -2\tau(1-\tau) \left[\frac{(1-\psi\tau)}{(1-\tau)} \sigma_i^2 + \frac{\sigma_{iP}}{A} \right] < 0. \end{aligned} \quad (20)$$

- ▶ After a couple of pages of algebra

$$\left. \frac{d\delta^* \left[\sigma_A^2(\tau, \psi), \mu_A(\tau, \psi) \right]}{d\psi} \right|_{d\tau=dK=0} \propto$$

$$2 \left[\mu_A(\tau, \psi) - K(1 - \tau) \right] \left[(1 - \tau\psi) \sigma_i^2 + \frac{\sigma_{iP}(1 - \tau)}{A} \right] - \sigma_A^2(\tau, \psi) \mu_i \geq 0 \quad (21)$$

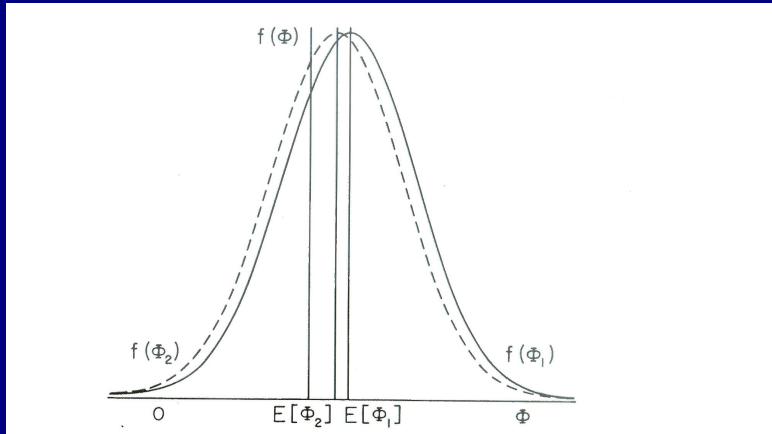
- Extending the results into mean and variance of the rate of return on equity

$$\left. \frac{d\mu_E(\tau, \psi)}{d\psi} \right|_{d\tau=0} = \frac{[\mu_A(\tau, \psi) - K(1 - \tau)]}{\left(1 - \delta^* \left[\sigma_A^2(\tau, \psi), \mu_A(\tau, \psi)\right]\right)^2} \frac{d\delta^* \left[\sigma_A^2(\tau, \psi), \mu_A(\tau, \psi)\right]}{d\psi} + \frac{1}{\left(1 - \delta^* \left[\sigma_A^2(\tau, \psi), \mu_A(\tau, \psi)\right]\right)} \frac{d\mu_A(\tau, \psi)}{d\psi} \geq 0 \quad (22)$$

- ▶ and

$$\begin{aligned} \frac{d\sigma_E^2(\tau, \psi)}{d\psi} \Big|_{d\tau=0} &= \frac{1}{\left(1 - \delta^* \left[\sigma_A^2(\tau, \psi), \mu_A(\tau, \psi)\right]\right)^2} \frac{d\sigma_A^2(\tau, \psi)}{d\psi} \\ &+ \frac{2\sigma_A^2(\tau, \psi)}{\left(1 - \delta^* \left[\sigma_A^2(\tau, \psi), \mu_A(\tau, \psi)\right]\right)^3} \frac{d\delta^* \left[\sigma_A^2(\tau, \psi), \mu_A(\tau, \psi)\right]}{d\psi} \end{aligned} \quad (23)$$

Change in Probability Density Function



Stochastic Optimal Control

- ▶ Ramirez, O.A., C.B. Moss, and W.G. Boggess. 1997. A Stochastic Optimal Control Formulation of the Consumption/Debt Decision. *Agricultural Finance Review* 47, 29-39.
- ▶ The basic idea of the model is to extend the proof of the optimal debt problem beyond the myopic-single period Collins's model.

- The starting point is to rewrite the rate of return formulation as a model of the change in equity

$$R_E = \frac{\frac{r_P}{A} + i - K\delta}{1 - \delta} = \frac{R_A - K\delta}{1 - \delta} \quad (24)$$

$$\Rightarrow dE(t) = \left[E(t) \frac{R_A(t) - K(t)\delta(t)}{1 - \delta(t)} - C(t) \right] dt$$

- Rewriting 24 as a stochastic differential equation – recognizing that the rate of return on assets is random

$$dE(t) = \left[E(t) \frac{R_A(t) - K(t)\delta(t)}{1 - \delta(t)} - C(t) \right] dt + \frac{E(t)\sigma_A(t)}{1 - \delta(t)} dz(t) \quad (25)$$

- Setting up the stochastic optimal control, we begin by simplifying Equation 25

$$\Delta E = \mu \Delta t + \sigma \Delta z \quad (26)$$

where Δz is a random increment – a Wiener process. Using Ito's lemma

$$\Delta z = \epsilon_t \sqrt{\Delta t} \Rightarrow \begin{cases} E[\Delta E] = \mu \Delta t \\ V[\Delta E] = \sigma^2 \Delta t \end{cases} . \quad (27)$$

- Applying these relationships to Equation 25

$$\begin{aligned} E[E(t)] &= \mu(t) = E(t) \frac{\mu_A(t) - K(t)\delta(t)}{1 - \delta(t)} - C(t) \\ V[E(t)] &= \frac{E(t)^2 \sigma_A^2(t)}{(1 - \delta(t))^2} dt \end{aligned} \quad (28)$$

- Using the power utility function

$$\max \delta(t), C(t) E \left[\int_0^\infty e^{-rt} \frac{C(t)^b}{b} dt \right] \quad (29)$$

subject to the equation of motion presented in Equation 25 .
 The resulting optimal paths can be written as

$$\delta^*(t) = 1 - \frac{(1-b) \sigma_A^2(t)}{\mu_A(t) - K(t)}$$

$$C^*(t) = E(t) D \quad (30)$$

$$\ni D = \frac{r - K(t)b}{1-b} - \frac{b(K(t) - \mu_A(t))^2}{2(1-b)^2 \sigma_A^2(t)}.$$