

Maximum Likelihood – Part II: Lecture XXVIII

Charles B. Moss ¹

¹University of Florida

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Readings

- * Cramer, J.S. 1986. *Econometric Applications of Maximum Likelihood Methods*. Cambridge University Press
- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition. (Chapter 14 pp. 509 598)

Differential Supply Systems

- Laitinen, K and H. Theil. 1978. Supply and Demand of the Multiproduct Firm. *European Economic Review* 11, 107 - 154.
- Basic system of input demand and supply equations

$$\begin{aligned}
 f_i d \ln(x_i) &= \gamma_1 \sum_{r=1}^m \theta_i^r g_r d \ln(y_r) + \sum_{j=1}^n \pi_{ij} d \ln(w_j) + \epsilon_i \\
 g_r d \ln(y_r) &= \sum_{s=1}^m \psi^* \theta_{rs}^* d \ln(p_s) - \sum_{s=1}^m \sum_{i=1}^n \psi^* \theta_{rs}^* \theta_i^s d \ln(w_i) + \epsilon_r^*
 \end{aligned}
 \tag{1}$$

Differential Supply System - Continued

- where f_i is the share of cost spent on input i and g_r is the share of revenue from output r

$$f_i = \frac{w_i x_i}{\sum_{j=1}^n w_j x_j} \quad g_r = \frac{p_r y_r}{\sum_{s=1}^m p_s y_s} \quad (2)$$

Where x_i is the level of input i purchased, w_i is the price of input i , y_r is the level of output r produced, and p_r is the price of that output.

- γ_1 , θ_i^r , π_{ij} , ψ^* , and θ_{rs}^* are estimated parameters.

Empirical Specification

- To implement this model, we use a first-difference approach (along the same lines as the Rotterdam model of demand)

$$\left[\frac{1}{2} (f_{it} + f_{i,t-1}) \right] \ln \left(\frac{x_{it}}{x_{i,t-1}} \right) = \bar{f}_{it} D x_{it}$$

$$\left[\frac{1}{2} (g_{it} + g_{i,t-1}) \right] \ln \left(\frac{y_{it}}{y_{i,t-1}} \right) = \bar{g}_{it} D y_{it} \quad (3)$$

$$\ln \left(\frac{w_{it}}{w_{i,t-1}} \right) = D w_{it} \quad \ln \left(\frac{p_{rt}}{p_{r,t-1}} \right) = D p_{rt}$$

Empirical Specification - Continued

- Incorporating this notation

$$\begin{aligned}\bar{f}_{it} Dw_{it} &= \sum_{r=1}^m \tilde{\theta}_i^r \bar{g}_{it} Dy_{rt} + \sum_{j=1}^n \pi_{ij} Dw_{jt} + \epsilon_{it} \\ \bar{g}_{rt} dy_{rt} &= \sum_{s=1}^m \tilde{\theta}_{rs}^* Dp_{st} - \sum_{s=1}^m \sum_{i=1}^n \tilde{\theta}_{rs}^* \tilde{\theta}_i^s Dw_{it} + \epsilon_{rt}^*\end{aligned}\tag{4}$$

Empirical Specification - Continued

- Using the Moss, Erickson, Ball and Mishra data we have two output (Livestock and Crops) and three inputs (Purchased Inputs, Labor, and Capital)

$$\begin{aligned}
 \bar{f}_{1t}Dw_{1t} &= \tilde{\theta}_1^1 \bar{g}_{1t}Dy_{1t} + \tilde{\theta}_2^2 \bar{g}_{2t}Dy_{2t} + \pi_{11}Dw_{1t} + \pi_{12}Dw_{2t} + \pi_{13}Dw_{3t} + \epsilon_{1t} \\
 \bar{f}_{2t}Dw_{2t} &= \tilde{\theta}_2^1 \bar{g}_{1t}Dy_{1t} + \tilde{\theta}_2^2 \bar{g}_{2t}Dy_{2t} + \pi_{12}Dw_{1t} + \pi_{22}Dw_{2t} + \pi_{23}Dw_{3t} + \epsilon_{2t} \\
 \bar{f}_{3t}Dw_{3t} &= \tilde{\theta}_3^1 \bar{g}_{1t}Dy_{1t} + \tilde{\theta}_3^2 \bar{g}_{2t}Dy_{2t} + \pi_{13}Dw_{1t} + \pi_{23}Dw_{2t} + \pi_{33}Dw_{3t} + \epsilon_{3t} \\
 \bar{g}_{1t}Dy_{1t} &= \tilde{\theta}_{11}^* Dp_{1t} + \tilde{\theta}_{12}^* Dp_{2t} - \left(\tilde{\theta}_{11}^* \tilde{\theta}_1^1 Dw_{1t} + \tilde{\theta}_{11}^* \tilde{\theta}_2^1 Dw_{2t} + \tilde{\theta}_{11}^* \tilde{\theta}_3^1 Dw_{3t} \right. \\
 &\quad \left. + \tilde{\theta}_{12}^* \tilde{\theta}_1^2 Dw_{1t} + \tilde{\theta}_{12}^* \tilde{\theta}_2^2 Dw_{2t} + \tilde{\theta}_{12}^* \tilde{\theta}_3^2 Dw_{3t} \right) + \epsilon_{1t}^* \\
 \bar{g}_{2t}Dy_{2t} &= \tilde{\theta}_{21}^* Dp_{1t} + \tilde{\theta}_{22}^* Dp_{2t} - \left(\tilde{\theta}_{21}^* \tilde{\theta}_1^1 Dw_{1t} + \tilde{\theta}_{21}^* \tilde{\theta}_2^1 Dw_{2t} + \tilde{\theta}_{21}^* \tilde{\theta}_3^1 Dw_{3t} \right. \\
 &\quad \left. + \tilde{\theta}_{22}^* \tilde{\theta}_1^2 Dw_{1t} + \tilde{\theta}_{22}^* \tilde{\theta}_2^2 Dw_{2t} + \tilde{\theta}_{22}^* \tilde{\theta}_3^2 Dw_{3t} \right) + \epsilon_{2t}^*
 \end{aligned} \tag{5}$$

Empirical Specification - Continued

- Imposing homogeneity and dropping the singular equation yields

$$\begin{aligned}\bar{f}_{1t}Dw_{1t} &= \tilde{\theta}_1^1 \bar{g}_{1t} Dy_{1t} + \tilde{\theta}_1^2 \bar{g}_{2t} Dy_{2t} + \pi_{11} Dw_{1t}^* + \pi_{12} Dw_{2t}^* + \epsilon_{1t} \\ \bar{f}_{2t}Dw_{2t} &= \tilde{\theta}_2^1 \bar{g}_{1t} Dy_{1t} + \tilde{\theta}_2^2 \bar{g}_{2t} Dy_{2t} + \pi_{12} Dw_{1t}^* + \pi_{22} Dw_{2t}^* + \epsilon_{2t} \\ \bar{g}_{1t}Dy_{1t} &= \tilde{\theta}_{11}^* \left(Dp_{1t} - \tilde{\theta}_1^1 Dw_{1t}^* - \tilde{\theta}_2^1 Dw_{2t}^* \right) + \tilde{\theta}_{12}^* \left(Dp_{2t} - \tilde{\theta}_1^2 Dw_{1t}^* - \tilde{\theta}_2^2 Dw_{2t}^* \right) + \epsilon_{1t}^* \\ \bar{g}_{2t}Dy_{2t} &= \tilde{\theta}_{21}^* \left(Dp_{1t} - \tilde{\theta}_1^1 Dw_{1t}^* - \tilde{\theta}_2^1 Dw_{2t}^* \right) + \tilde{\theta}_{22}^* \left(Dp_{2t} - \tilde{\theta}_1^2 Dw_{1t}^* - \tilde{\theta}_2^2 Dw_{2t}^* \right) + \epsilon_{2t}^*\end{aligned}\quad (6)$$

- As a starting point, consider estimating the first two equations using ordinary least squares.

Ordinary Least Squares Results

Independent Variable	$\bar{f}_{1t}Dw_{1t}$	$\bar{f}_{2t}Dw_{2t}$
$\bar{g}_{1t}Dy_{1t}$	0.1816 (0.0725)	0.1847 (0.0642)
$\bar{g}_{2t}Dy_{2t}$	0.5124 (0.0564)	-0.0232 (0.0499)
Dw_{1t}^*	-0.1152 (0.0610)	0.0995 (0.0540)
Dw_{2t}^*	-0.0772 (0.0459)	-0.2253 (0.0406)

Maximum Likelihood Estimates

- Using the Multiple Equation Results from Lecture 27

$$\nabla_{\theta} L(\theta) = \frac{1}{2} \sum_{t=1}^T \epsilon_{1t} \Omega^{-1} Z_t$$

$$Z_t = \begin{bmatrix} -\bar{g}_{1t} Dy_{1t} & -\bar{g}_{2t} Dy_{2t} & -Dw_{1t}^* & -Dw_{2t}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & -Dw_{1t}^* & -\bar{g}_{1t} Dy_{1t} & -\bar{g}_{2t} y_{2t} & -Dw_{2t}^* \end{bmatrix} \quad (7)$$

$$\nabla_{\theta\theta}^2 L(\theta) = -\frac{1}{2} \sum_{t=1}^T Z_t' \Omega^{-1} Z_t \quad (8)$$

OLS and ML Results

Coefficient	OLS	ML
$\tilde{\theta}_1^1$	0.18864 (0.00458)	0.20848 (0.00255)
$\tilde{\theta}_1^2$	0.51338 (0.00277)	0.51512 (0.00157)
π_{11}	-0.13016 (0.00321)	-0.19230 (0.00137)
π_{12}	-0.01330 (0.00117)	-0.00233 (0.00060)
$\tilde{\theta}_2^1$	0.22050 (0.00438)	0.21826 (0.00198)
$\tilde{\theta}_2^2$	-0.02012 (0.00277)	-0.02012 (0.00125)
π_{22}	-0.21029 (0.00180)	-0.17803 (0.00065)

Slightly Nonlinear

- Dropping the last equation (again because of singularity) yields

$$\begin{aligned}\bar{f}_{1t}Dw_{1t} &= \tilde{\theta}_1^1\bar{g}_{1t}Dy_{1t} + \tilde{\theta}_1^2\bar{g}_{2t}Dy_{2t} + \pi_{11}Dw_{1t}^* + \pi_{12}Dw_{2t}^* + \epsilon_{1t} \\ \bar{f}_{2t}Dw_{2t} &= \tilde{\theta}_2^1\bar{g}_{1t}Dy_{1t} + \tilde{\theta}_2^2\bar{g}_{2t}Dy_{2t} + \pi_{12}Dw_{1t}^* + \pi_{22}Dw_{2t}^* + \epsilon_{2t} \\ \bar{g}_{1t}Dy_{1t} &= \tilde{\theta}_{11}^* \left(Dp_{1t} - \tilde{\theta}_1^1Dw_{1t}^* - \tilde{\theta}_1^2Dw_{2t}^* \right) + \tilde{\theta}_{12}^* \left(Dp_{2t} - \tilde{\theta}_2^1Dw_{1t}^* - \tilde{\theta}_2^2Dw_{2t}^* \right) + \epsilon_{1t}^*\end{aligned}\quad (9)$$

- The error term for Maximum Likelihood becomes

$$\begin{aligned}\epsilon_{1t} &= \bar{f}_{1t}Dw_{1t} - \left(\tilde{\theta}_1^1\bar{g}_{1t}Dy_{1t} + \tilde{\theta}_1^2\bar{g}_{2t}Dy_{2t} + \pi_{11}Dw_{1t}^* + \pi_{12}Dw_{2t}^* \right) \\ \epsilon_{2t} &= \bar{f}_{2t}Dw_{2t} - \left(\tilde{\theta}_2^1\bar{g}_{1t}Dy_{1t} + \tilde{\theta}_2^2\bar{g}_{2t}Dy_{2t} + \pi_{12}Dw_{1t}^* + \pi_{22}Dw_{2t}^* \right) \\ \epsilon_{1t}^* &= \bar{g}_{1t}Dy_{1t} - \left(\tilde{\theta}_{11}^* \left(Dp_{1t} - \tilde{\theta}_1^1Dw_{1t}^* - \tilde{\theta}_1^2Dw_{2t}^* \right) + \tilde{\theta}_{12}^* \left(Dp_{2t} - \tilde{\theta}_2^1Dw_{1t}^* - \tilde{\theta}_2^2Dw_{2t}^* \right) \right)\end{aligned}\quad (10)$$

Slightly Nonlinear - Continued

- Taking the gradient

$$[\nabla_{\theta} L(\theta)]' = \begin{pmatrix} -\bar{g}_{1t} Dy_{1t} & 0 & \tilde{\theta}_{11}^* Dw_{1t}^* & \\ -\bar{g}_{2t} Dy_{2t} & 0 & \tilde{\theta}_{12}^* Dw_{1t}^* & \\ -Dw_{1t}^* & 0 & 0 & \\ -Dw_{2t}^* & -Dw_{1t}^* & 0 & \\ 0 & -\bar{g}_{1t} Dy_{1t} & \tilde{\theta}_{11}^* Dw_{2t}^* & \\ 0 & -\bar{g}_{2t} Dy_{2t} & \tilde{\theta}_{12}^* Dw_{2t}^* & \\ 0 & 0 & -Dp_{1t} + \tilde{\theta}_1^1 Dw_{1t}^* + \tilde{\theta}_2^1 Dw_{2t}^* & \\ 0 & 0 & -Dp_{2t} + \tilde{\theta}_1^2 Dw_{1t}^* + \tilde{\theta}_2^2 Dw_{2t}^* & \end{pmatrix} \sim \begin{bmatrix} \tilde{\theta}_1^1 \\ \tilde{\theta}_1^2 \\ \pi_{11} \\ \pi_{12} \\ \tilde{\theta}_2^1 \\ \tilde{\theta}_2^2 \\ \tilde{\theta}_{11}^* \\ \tilde{\theta}_{12}^* \end{bmatrix} \quad (11)$$

Readings

- * Cramer, J.S. 1986. *Econometric Applications of Maximum Likelihood Methods*. Cambridge University Press
- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition. (Chapter 14 pp. 509 598)