

Generalized Method of Moments – Part I: Lecture XXV

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Readings

- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition. (Chapter 13 pp. 455 507)
- * Hall, A. 2005. *Generalized Method of Moments*. Oxford University Press (Chapters 1 3 pp. 1 114)

Basic Derivation of the Linear Estimator

- Starting with the basic linear model

$$y_t = x_t' \theta_0 + u_t \quad (1)$$

where y_t is the dependent variable, x_t is the vector of independent variables, θ_0 is the parameter vector, and u_t is the residual. In addition to these variables we will introduce the notion of a vector of instrumental variables denoted z_t .

- Reworking the original formulation slightly, we can express the residual as a function of the parameter vector

$$u_t(\theta_0) = y_t - x_t' \theta_0 \quad (2)$$

- Based on this expression, estimation follows from the population moment condition

$$E[z_t u_t(\theta_0)] = 0 \quad (3)$$

Or more specifically, we select the vector of parameters so that the residuals are orthogonal to the set of instruments.

- Note the similarity between these conditions and the orthogonality conditions implied by the linear projection space

$$P_c = X (X'X)^{-1} X' \quad (4)$$

- Further developing the orthogonality condition, note that if a single θ_0 solves the orthogonality conditions, or that θ_0 is unique that

$$E[z_t u_t(\theta)] = 0 \text{ if and only if } \theta = \theta_0 \quad (5)$$

Alternatively

$$E[z_t u_t(\theta)] \neq 0 \text{ if } \theta \neq \theta_0 \quad (6)$$

- Going back to the original formulation

$$E[z_t u_t(\theta)] = E[z_t (y_t - x_t' \theta)] \quad (7)$$

- Taking the first-order Taylor series expansion

$$\begin{aligned} E[z_t (y_t - x_t' \theta)] &= E[z_t (y_t - x_t' \theta_0)] - E[z_t x_t'] (\theta - \theta_0) \\ &\Leftarrow \frac{\partial}{\partial \theta} (y_t - x_t' \theta) = -x_t' \end{aligned} \quad (8)$$

- Given that $E[z_t(y_t - x_t'\theta_0)] = E[z_t u_t(\theta_0)] = 0$, this expression implies

$$E[z_t(y_t - x_t'\theta)] = E[z_t x_t'](\theta_0 - \theta) \quad (9)$$

- Given this background, the most general form of the minimand (objective function) of the GMM model ($Q_t(\theta)$) can be expressed as

$$Q_T(\theta) = \left\{ \frac{1}{T} u(\theta)' Z \right\} W_T \left\{ \frac{1}{T} Z' u(\theta) \right\} \quad (10)$$

where T is the number of observations, $u(\theta)$ is a column vector of residuals, Z is a matrix of instrumental variables, and W_T is a weighting matrix (akin to a variance matrix).

- Given that W_T is a type of variance matrix, it is positive definite guaranteeing that

$$z' W_T z > 0 \quad (11)$$

for any vector z .

- Building on the initial model

$$E[z_t u_t(\theta)] = \frac{1}{T} Z' u(\theta) \quad (12)$$

In the linear case

$$E[z_t u_t(\theta)] = \frac{1}{T} Z' (y - X\theta) \quad (13)$$

- Given that W_T is positive definite and the optimality condition when the residuals are orthogonal to the variances based on the parameters

$$E[z_t u_t(\theta_0)] \Rightarrow \frac{1}{T} Z' u(\theta_0) = 0 \Rightarrow Q_T(\theta_0) = 0 \quad (14)$$

- Working the minimization problem out for the linear case

$$\begin{aligned}
 Q_T(\theta) &= \frac{1}{T^2} [(y - X\theta)' Z] W_T [Z' (y - X\theta)] \\
 &= \frac{1}{T^2} [y' Z W_T - \theta' X' Z W_T] [Z' y - Z' X \theta] \\
 &= \frac{1}{T^2} [y' Z W_T Z' y - \theta' X' Z W_T Z' y - y' Z W_T Z' X \theta + \theta' X' Z W_T Z' X \theta]
 \end{aligned} \tag{15}$$

Note that since $Q_T(\theta)$ is a scalar,

$$\theta' X' Z W_T Z' y = y' Z W_T Z' X \theta \tag{16}$$

are scalars so

$$Q_T(\theta) = \frac{1}{T^2} [y' Z W_T Z' y + \theta' X' Z W_T Z' X \theta - 2\theta' X' Z W_T Z' y] \tag{17}$$

- Solving the first-order conditions

$$\begin{aligned}\nabla_{\theta} Q_T(\theta) &= \frac{1}{T^2} [2X'ZW_TZ'X\theta - 2X'ZW_TZ'y] = 0 \\ \Rightarrow \hat{\theta} &= (X'ZW_TZ'X)^{-1} (X'ZW_TZ'y)\end{aligned}\tag{18}$$

- An alternative approach is to solve the implicit first-order conditions above. Starting with

$$\begin{aligned}
 \nabla_{\theta} Q_T(\theta) &= \frac{1}{T^2} [2X'ZW_TZ'X\theta - 2X'ZW_TZ'y] = 0 \\
 \Rightarrow \frac{1}{2}\nabla_{\theta} Q_T(\theta) &= \left(\frac{1}{T}X'Z\right) W_T \left(\frac{1}{T}Z'X\theta\right) - \\
 &\quad \left(\frac{1}{T}X'Z\right) W_T \left(\frac{1}{T}Z'y\right) = 0 \\
 &= \left(\frac{1}{T}X'Z\right) W_T \left(\frac{1}{T}Z'X\theta - Z'y\right) = 0 \\
 &= \left(\frac{1}{T}X'Z\right) W_T \left(\frac{1}{T}Z' \{y - X\theta\}\right) = 0 \\
 \Rightarrow -\frac{1}{2}\nabla_{\theta} Q_T(\theta) &= \left(\frac{1}{T}X'Z\right) W_T \left(\frac{1}{T}Z' \{y - X\theta\}\right) = 0 \\
 &= \left(\frac{1}{T}X'Z\right) W_T \left(\frac{1}{T}Z'u(\theta)\right) = 0
 \end{aligned} \tag{19}$$

The Limiting Distribution

- By the Central Limit Theory

$$\frac{1}{\sqrt{T}} Z' u(\theta) = \frac{1}{\sqrt{T}} \sum_{t=1}^T z_t u_t(\theta) \xrightarrow{d} N(0, S) \quad (20)$$

Therefore

$$\begin{aligned} \frac{1}{\sqrt{T}} (\hat{\theta} - \theta_0) &\xrightarrow{d} N(0, MSM') \\ M &= (E[x_t z_t'] W E[z_t x_t'])^{-1} E[x_t z_t'] W \\ S &= \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{s=1}^T \sum_{t=1}^T E[u_t u_s z_t z_s'] = E[u^2 z z'] \\ MSM' &= \left\{ E[z_t x_t'] \right\}^{-1} S \left\{ E[x_t z_t'] \right\} \end{aligned} \quad (21)$$

- Under the classical instrumental variable assumptions

$$\hat{S}_T = \frac{1}{T} \sum_{t=1}^T \hat{u}_t^2 z_t z_t'$$

(22)

$$\hat{S}_{CIV} = \frac{\hat{\sigma}_T^2}{T} Z' Z$$

Example: Differential Demand Model

- Following Theil's model for the derived demand for inputs

$$f_i d \ln [q_i] = \theta_i d \ln [O] + \sum_{j=1}^n \pi_{ij} d \ln [p_j] + \epsilon_i \quad (23)$$

where f_i is the factor share of input i ($f_i = p_i q_i / C$ such that p_i is the price of the input, q_i is the level of the input used, and C is the total cost of production), O is the level of output, and ϵ_i is the residual. The model is typically estimated as

$$\bar{f}_{it} D [q_{it}] = \theta_i D [O_t] + \sum_{j=1}^n \pi_{ij} D [p_{jt}] + \epsilon_t \quad (24)$$

such that $\bar{f}_{it} = \frac{1}{2} (f_{it} + f_{i,t-1})$ and $D [x_t] = \ln [x_t] - \ln [x_{t-1}]$.

- Applying this to capital in agricultural from Jorgensons database, the output is an index of all outputs and the inputs are capital (p_{ct}), labor (p_{lt}), energy (p_{et}), and materials (p_{mt}).
 - Thus,

$$X = \begin{bmatrix} O_t & p_{ct} & p_{lt} & p_{et} & p_{mt} \end{bmatrix}_{t=1}^T$$

$$Z = \begin{bmatrix} O_t & p_{ct} & p_{lt} & p_{et} & p_{mt} & O_t^2 & p_{ct}^2 & p_{lt}^2 & p_{et}^2 & p_{mt}^2 \end{bmatrix}_{t=1}^T \quad (25)$$

- Rewriting the demand model

$$y = X\theta \quad (26)$$

- Thus, the objective function for the Generalized Method of Moments estimator is

$$Q_T(\theta) = (y - X\theta)' ZW_T Z' (y - X\theta) \quad (27)$$

- Initially we let $W_T = I$ and minimize $Q_T(\theta)$. This yields a first approximation to the estimates in the second column of Table 1.
- Updating W_T

$$\hat{S}_{CIV} = \frac{\hat{\sigma}_T^2}{T} Z'Z \quad (28)$$

and resolving yield the second stage GMM estimates in the second column of Table 1

Table: Generalized Methods of Moments Estimates of Differential Demand Equation

Parameter	First Stage GMM	Second Stage GMM	Ordinary Least Squares
Output	0.01588 (0.00865)	0.01592 (0.00825)	0.01591 (0.00885)
Capital	-0.00661 (0.00280)	-0.00675 (0.00261)	-0.00675 (0.00280)
Labor	0.00068 (0.03429)	0.00058 (0.00334)	0.00058 (0.00359)
Energy	0.00578 (0.00434)	0.00572 (0.00402)	0.00572 (0.00432)
Materials	0.02734 (0.01215)	0.02813 (0.01068)	0.02813 (0.01146)