

Random Effect Models for Panel Data: Lecture XXIV

Charles B. Moss ¹

¹University of Florida

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Readings

- Arellano, M. 2003. *Panel Data Econometrics*. Oxford University Press. (Chapters 3 and 4 pp. 31-54).
- * Cameron, A. C. and P. K. Trivedi. 2010. *Microeconometrics Using Stata* Revised Edition. Stata Press. (Chapter 8 [Section 8.3 - 8.7] pp. 240- 253).
- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition. (Chapter 11 Section 11.5 pp. 359 364)
- * Hsiao, C. 1990. *Analysis of Panel Data*. Cambridge University Press. (Chapters 3 [Section 3.4-3.9] pp. 32-68)

Sweeping the Data

- In Lecture 8, we discussed the **sweep** operation for the covariance matrices.
- Now, we need to develop it in the context of the **covariance estimator** of the **fixed effect model**.
- Returning to the fixed effect model

$$\begin{aligned}
 \hat{\alpha}_i^* &= \bar{y}_i - \beta' \bar{x}_i \quad i = 1, \dots, N \\
 \bar{y}_i &= \frac{1}{T} \sum_{t=1}^T y_{it} \quad \bar{x}_i = \frac{1}{T} \sum_{t=1}^T x_{it} \\
 \hat{\beta}_{CV} &= \left[\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i) (x_{it} - \bar{x}_i)' \right] \left[\sum_{i=1}^N \sum_{t=1}^T (x_{it} - \bar{x}_i) (y_{it} - \bar{y}_i) \right]
 \end{aligned} \tag{1}$$

Sweep Matrix

- Defining the basic sweep matrix

$$Q = I_T - \frac{1}{T} ee'$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 - \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 1 - \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 - \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 - \frac{1}{4} \end{bmatrix} \quad (2)$$

- Note that this matrix is idempotent

$$QQ = I_T - 2\frac{1}{T}ee' + \frac{1}{T^2}ee'ee' \ni e'e \equiv T$$

$$\Rightarrow QQ = I_T - 2\frac{1}{T}ee' + \frac{1}{T}ee' = Q \quad (3)$$

Fixed Effect Regression

- At the most basic level, the sweep matrix simplifies the estimation of the individual covariance matrices

$$W_{XX,i} = X_i' Q X_i \quad W_{XY,i} = X_i' Q y_i \quad (4)$$

- On a slightly grander scale, we can envision doing the entire fixed effect estimation in one operation.
 - Imagine stacking the X_i matrices $X = [X_1 \quad X_2 \quad \cdots \quad X_N]'$.
 - The **fixed effects regression** simply becomes

$$\hat{\beta}_{CV} = (X' [I_{N \times N} \otimes Q] X)^{-1} (X' [I_{N \times N} \otimes Q] \text{vec}(y)) \quad (5)$$

Random Effects Model

- Regression analysis typically assumes that a large number of factors affect the value of the dependent variable, while some of the variables are measured directly in the model the remaining variables can be summarized by a random distribution

$$\begin{aligned} y_{it} &= \alpha + \beta x_{it} + \gamma z_{it} + \epsilon_{it} \\ &= (\alpha + \gamma E[z_{it}]) + \beta x_{it} + (\epsilon_{it} + [z_{it} - E[z_{it}]]) \end{aligned} \quad (6)$$

- When “numerous observations” on are observed over time, it is assumed that some of the omitted variables represent factors peculiar to individual and time periods.

Panel Specification

- Going back to the panel specification

$$\begin{aligned}y_{it} &= \alpha^* + \beta' x_{it} + \nu_{it} \\ \nu_{it} &= \alpha_i + \lambda_t + \mu_{it}\end{aligned}\tag{7}$$

- α_i is the “individual variation” in the constant,
- λ_t is the “time variation” in the constant, and
- μ_{it} is the “pure error term”.

Identification

- The expected value of each of these components is equal to zero

$$E[\alpha_i] = E[\lambda_t] = E[u_{it}] = 0 \quad (8)$$

- We also assume that each of these error terms are uncorrelated

$$E[\alpha_i \lambda_t] = E[\alpha_i u_{it}] = E[\lambda_t u_{it}] = 0 \quad (9)$$

- The variance of the “individual effects” is then defined as

$$E[\alpha_i \alpha_j] = \begin{cases} \sigma_\alpha^2 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (10)$$

Identification

- The variance for the “time effects” is then defined as

$$E [\lambda_t \lambda_s] = \begin{cases} \sigma_\lambda^2 & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases} \quad (11)$$

- The variance for the “pure residuals” are them

$$E [u_{it} u_{js}] = \begin{cases} \sigma_u^2 & \text{if } i = j, t = s \\ 0 & \text{Otherwise} \end{cases} \quad (12)$$

- Finally, we assume that each of these effects are uncorrelated with the independent variables

$$E [\alpha_i x'_{it}] = E [\lambda_t x'_{it}] = E [u_{it} x'_{it}] = 0 \quad (13)$$

Mother of all Heteroscedasticity

- The variance of y_{it} on x_{it} based on the assumption above is

$$\sigma_y^2 = \sigma_\alpha^2 + \sigma_\lambda^2 + \sigma_u^2 \quad (14)$$

- Thus, this kind of model is typically referred to as a variance-component (or error-components) model.
- Letting

$$\begin{aligned} \tilde{X}_i &= (e, X_i) \\ \delta &= (\mu, \beta') \\ \nu_i' &= (\nu_{i1}, \dots, \nu_{iT}) \\ \nu_{it} &= \alpha_i + \mu_{it} \end{aligned} \quad (15)$$

the panel estimation model can be written in vector form as

$$y_i = \tilde{X}_i \delta + \nu_i \quad i = 1, \dots, N \quad (16)$$

Variance Matrix

- The expected value of the residual becomes

$$\begin{aligned} E[\nu_i \nu_i'] &= \sigma_u^2 I_T + \sigma_\alpha^2 e e' = V \\ V^{-1} &= \frac{1}{\sigma_u^2} \left[I_T - \frac{\sigma_\alpha^2}{\sigma_u^2 + T \sigma_\alpha^2} \right] \end{aligned} \quad (17)$$

- Using the basic covariance estimator

$$\begin{aligned} Qy_i &= Qe\mu + QX_i\beta + Qe\alpha_i + Qu_i \\ &= QX_i\beta + Qu_i \end{aligned} \quad (18)$$

- Whether α_i is fixed or random, the covariance estimator is unbiased.
- However, if α_i is random, the covariance estimator is not the best linear unbiased estimator (BLUE).
- Instead, a BLUE estimator can be derived using generalized least squares (GLS).

Generalized Least Squares Estimator

- Because both u_{it} and u_{is} contain α_i , they are correlated (by the individual effect)

$$\delta = (\mu, \beta')$$

$$\left[\sum_{i=1}^N \tilde{X}_i' V^{-1} \tilde{X}_i \right] \hat{\delta}_{GLS} = \left[\sum_{i=1}^N \tilde{X}_i' V^{-1} y_i \right] \quad (19)$$

- Inverse Variance Matrix

$$V^{-1} = \frac{1}{\sigma_u^2} \left[\left(I_T - \frac{1}{T} ee' \right) + \psi \frac{1}{T} ee' \right] = \frac{1}{\sigma_u^2} \left[Q + \psi \frac{1}{T} ee' \right]$$

$$\psi = \frac{\sigma_u^2}{\sigma_u^2 + T\sigma_\alpha^2} \quad (20)$$

Procedure for Estimation

- Starting specification

$$\left[W_{\tilde{X}\tilde{X}} + \psi B_{\tilde{X}\tilde{X}} \right] \begin{bmatrix} \hat{\mu} \\ \hat{\beta} \end{bmatrix}_{GLS} = \left[W_{\tilde{X}y} + \psi B_{\tilde{X}y} \right] \quad (21)$$

- Defining

$$\begin{aligned} T_{\tilde{X}\tilde{X}} &= \sum_{i=1}^N \tilde{X}_i' \tilde{X}_i & T_{\tilde{X}y} &= \sum_{i=1}^N \tilde{X}_i' y_i \\ B_{\tilde{X}\tilde{X}} &= \frac{1}{T} \sum_{i=1}^N \left(\tilde{X}_i' e e' \tilde{X}_i \right) & B_{\tilde{X}y} &= \frac{1}{T} \sum_{i=1}^N \left(\tilde{X}_i' e e' y_i \right) \\ W_{\tilde{X}\tilde{X}} &= T_{\tilde{X}\tilde{X}} - B_{\tilde{X}\tilde{X}} & W_{\tilde{X}y} &= T_{\tilde{X}y} - B_{\tilde{X}y} \end{aligned} \quad (22)$$

(Hsiao p. 35)

- Continued

- The matrices $B_{\tilde{X}\tilde{X}}$ and $B_{\tilde{X}y}$ contain the sum of squares and the sum of cross products between groups,
- $W_{\tilde{X}\tilde{X}}$ and $W_{\tilde{X}y}$ are the corresponding matrices within groups, and
- $T_{\tilde{X}\tilde{X}}$ and $T_{\tilde{X}y}$ are the corresponding matrices for total variation.

Help Me!

- This looks bad, but think about

$$\begin{aligned} (X'QX)\beta &= (X'Qy) \\ \left[X' \left(I_T - \frac{1}{T} ee' \right) X \right] \beta &= X' \left(I_T - \frac{1}{T} ee' \right) y \\ \left[X'X - \frac{1}{T} X' ee' X \right] \beta &= X'y - \frac{1}{T} X' ee' y \end{aligned} \quad (23)$$

- With

$$\begin{aligned} T_{\tilde{X}\tilde{X}} &\Rightarrow X'X & B_{\tilde{X}\tilde{X}} &\Rightarrow \frac{1}{T} X' ee' X \\ W_{\tilde{X}y} &\Rightarrow X'y & B_{\tilde{X}y} &\Rightarrow \frac{1}{T} X' ee' y \end{aligned} \quad (24)$$

System Specification

- Solving this system yields

$$\begin{bmatrix} \psi NT & \psi T \sum_{i=1}^N \bar{x}_i' \\ \psi T \sum_{i=1}^N \bar{x}_i & \sum_{i=1}^N X_i' Q X_i + \psi T \sum_{i=1}^N \bar{x}_i \bar{x}_i' \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\beta} \end{bmatrix}_{GLS} = \begin{bmatrix} \psi NT \bar{y} \\ \sum_{i=1}^N X_i' Q y_i + \psi T \sum_{i=1}^N \bar{x}_i \bar{y}_i \end{bmatrix} \quad (25)$$

- Using the inverse of a partitioned matrix

$$\begin{aligned}\hat{\beta}_{GLS} &= \left[\frac{1}{T} \sum_{i=1}^N X_i' Q X_i + \psi \sum_{i=1}^N (\bar{x}_i - \bar{x})(\bar{x}_i - \bar{x})' \right]^{-1} \\ &\times \left[\frac{1}{T} \sum_{i=1}^N X_i' Q y_i + \psi \sum_{i=1}^N (\bar{x}_i - \bar{x})(\bar{y}_i - \bar{y}) \right] \quad (26) \\ &= \Delta \hat{\beta}_b + (I_K - \Delta) \hat{\beta}_{CV} \\ \hat{\mu}_{GLS} &= \bar{y} - \hat{\beta}_{GLS} \bar{x}\end{aligned}$$

- Where

$$\Delta = \psi T \left[\sum_{i=1}^N X_i' Q X_i + \psi T \sum_{i=1}^N (\bar{x}_i - \bar{x}) (\bar{x}_i - \bar{x})' \right]^{-1} \\ \times \left[\sum_{i=1}^N (\bar{x}_i - \bar{x}) (\bar{y}_i - \bar{y}) \right] \\ \hat{\beta}_b = \left[\sum_{i=1}^N (\bar{x}_i - \bar{x}) (\bar{x}_i - \bar{x})' \right]^{-1} \left[\sum_{i=1}^N (\bar{x}_i - \bar{x}) (\bar{y}_i - \bar{y}) \right] \quad (27)$$

- Where β_b is the between estimator.

- The variance of the estimator can be written as

$$V\left(\hat{\beta}_{GLS}\right) = \sigma_u^2 \left[\sum_{i=1}^N X_i' Q X_i + T\psi \sum_{i=1}^N (\bar{x}_i - \bar{x})(\bar{x}_i - \bar{x})' \right]^{-1} \quad (28)$$

- Given that we don't know ψ *a priori*, we estimate

$$\hat{\sigma}_u^2 = \frac{\sum_{i=1}^N \sum_{t=1}^T \left[(y_{it} - \bar{y}_i) - \hat{\beta}'_{CV} (x_{it} - \bar{x}_i) \right]^2}{N(T-1) - K} \quad (29)$$

$$\hat{\sigma}_\alpha^2 = \frac{\sum_{i=1}^N (\bar{y}_i - \bar{\mu} - \bar{\beta}\bar{x}_i)^2}{N - (K+1)} - \frac{1}{T} \hat{\sigma}_u^2$$

Readings – Next Time

- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition. (Chapter 13 pp. 455 507)
- * Hall, A. 2005. *Generalized Method of Moments*. Oxford University Press (Chapters 1 3 pp. 1 114)