

Seemingly Unrelated Regressions - Introduction to Systems: Lecture XX

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Readings

- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition (Chapter 10 Section 10.1 10.5 pp. 290 313)
- * Theil, H. 1971. *Principles of Econometrics*. John Wiley & Sons. (Chapter 7).

Systems of Equations in Economics

- There are two general types of systems of equations in econometrics:
 - Simultaneous Equations – where the values of some variables are determined by the interaction of two or more equations. For example, supply and demand for a good

$$\begin{aligned}q_1^s &= \alpha_0 + \alpha_1 p_1 + \alpha_2 w_1 + \alpha_3 w_2 + \epsilon_s \\q_1^d &= \beta_0 + \beta_1 p_1 + \beta_2 p_2 + \epsilon_d\end{aligned}\tag{1}$$

solving the system given the equilibrium $q_1^s = q_1^d = q_1$, in this case the quantity and price are simultaneously determined

$$\begin{aligned}p_1 &= \frac{(-\alpha_0 + \beta_0)}{(\alpha_1 - \beta_1)} - \frac{\alpha_2}{(\alpha_1 - \beta_1)} w_1 - \frac{\alpha_3}{(\alpha_1 - \beta_1)} w_2 \\&\quad + \frac{\beta_2}{(\alpha_1 - \beta_1)} p_2 + \frac{(-\epsilon_s + \epsilon_d)}{(\alpha_1 - \beta_1)}\end{aligned}\tag{2}$$

- Seemingly Unrelated Regression – where the values of one equation do not affect the other equation.

- We will develop the two primary approaches to dealing with simultaneous equations after break:
 - Two-Stage Least Squares – This approach involves estimating a reduced form equation (i.e., the linear specification for price in Equation 2) and then substituting the estimated values into the original equation or equations

$$\begin{aligned}q_1^s &= \alpha_0 + \alpha_1 \hat{p}_1 + \alpha_2 w_1 + \alpha_3 w_2 + \epsilon_s \\ q_1^d &= \beta_0 + \beta_1 \hat{p}_1 + \beta_2 p_2 + \epsilon_d\end{aligned}\tag{3}$$

- Generalized Instrumental Variable Estimation – define a set of variables that explain the variation in endogenous variable. For example, if we wanted to estimate the demand equation from 1, we could use the information from the supply equation
 $Z = \begin{bmatrix} 1 & w_1 & w_2 \end{bmatrix}$

$$\begin{aligned}\beta_{IV} &= (X' P_Z X)^{-1} (X' P_Z y) \\ P_Z &= Z (Z' Z)^{-1} Z'\end{aligned}\tag{4}$$

Seemingly Unrelated Regression

- Seemingly Unrelated Regression is really a heteroscedasticity problem where the errors between equations are correlated.
- As a starting point, consider a quadratic cost function

$$\begin{aligned}
 C = \alpha_0 + & \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}' \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}' \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\
 & + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}' \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
 & + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}' \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
 \end{aligned}
 \tag{5}$$

Seemingly Unrelated Regression - Continued

- Applying Shephard's Lemma, the demand for each input is equal to the derivative of the cost function with respect to each input price

$$x_1(w_1, w_2, w_3, y_1, y_2) = \frac{\partial C}{\partial w_1} = \alpha_1 + A_{11}w_1 + A_{12}w_2 + A_{13}w_3 + \Gamma_{11}y_1 + \Gamma_{21}y_2$$

$$x_2(w_1, w_2, w_3, y_1, y_2) = \frac{\partial C}{\partial w_2} = \alpha_2 + A_{12}w_1 + A_{22}w_2 + A_{23}w_3 + \Gamma_{12}y_1 + \Gamma_{22}y_2$$

$$x_3(w_1, w_2, w_3, y_1, y_2) = \frac{\partial C}{\partial w_3} = \alpha_3 + A_{13}w_1 + A_{23}w_2 + A_{33}w_3 + \Gamma_{13}y_1 + \Gamma_{23}y_2 \quad (6)$$

Seemingly Unrelated Regression - Continued

- Because of the summing up restriction (i.e., since $w_1x_1 + w_2x_2 + w_3x_3 \equiv C$) we estimate a system of three equations – the cost function and two derived demand equations.

$$\begin{aligned}
 C = & \alpha_0 + \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 + A_{11} \frac{w_1^2}{2} + A_{12} w_1 w_2 + A_{13} w_1 w_3 + \\
 & A_{22} \frac{w_2^2}{2} + A_{23} w_2 w_3 + A_{33} \frac{w_3^2}{2} + \beta_1 y_1 + \beta_2 y_2 + B_{11} \frac{y_1^2}{2} + \\
 & B_{12} y_1 y_2 + B_{22} \frac{y_2^2}{2} + \Gamma_{11} w_1 y_1 + \Gamma_{21} w_1 y_2 + \Gamma_{12} w_2 y_1 + \Gamma_{22} w_2 y_2 \\
 & \Gamma_{13} w_3 y_1 + \Gamma_{23} w_3 y_2 + \epsilon_1
 \end{aligned} \tag{7}$$

$$x_1 = \alpha_1 + A_{11} w_1 + A_{12} w_2 + A_{13} w_3 + \Gamma_{11} y_1 + \Gamma_{21} y_2 + \epsilon_2$$

$$x_2 = \alpha_2 + A_{12} w_1 + A_{22} w_2 + A_{23} w_3 + \Gamma_{12} y_1 + \Gamma_{22} y_2 + \epsilon_3$$

Seemingly Unrelated Regression - Continued

- We want to estimate these equations in a system for two reason:
 - It is likely that the random outcome of one equation will affect the outcome of another equation

$$V \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \neq \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \quad (8)$$

- Second, we are really interested in the cross-equation restrictions so that the parameters of the factor demand equations are the same of the parameters of the cost function.

Consumer Demand Equation

- As a starting point, consider the Rotterdam demand system that we have used in the preceding homeworks

$$\begin{aligned}\bar{w}_{1t} d \ln (q_{1t}) &= \theta_1 d \ln (Q_t) + \pi_{11} d \ln (p_{1t}) + \pi_{12} d \ln (p_{2t}) + \pi_{13} d \ln (p_{3t}) + \epsilon_{1t} \\ \bar{w}_{2t} d \ln (q_{2t}) &= \theta_2 d \ln (Q_t) + \pi_{12} d \ln (p_{1t}) + \pi_{22} d \ln (p_{2t}) + \pi_{23} d \ln (p_{3t}) + \epsilon_{2t} \\ \bar{w}_{3t} d \ln (q_{3t}) &= \theta_3 d \ln (Q_t) + \pi_{13} d \ln (p_{1t}) + \pi_{23} d \ln (p_{2t}) + \pi_{33} d \ln (p_{3t}) + \epsilon_{3t}\end{aligned}\quad (9)$$

- Ignoring for the moment the symmetry restrictions, we could envision estimating this system of equations using the stacked regression approach from our discussion of Kronecker products

$$\begin{aligned}x_t &= \begin{bmatrix} d \ln (Q_t) & d \ln (p_{1t}) & d \ln (p_{2t}) & d \ln (p_{3t}) \end{bmatrix} \\ y_t &= \begin{bmatrix} \bar{w}_{1t} d \ln (q_{1t}) & \bar{w}_{2t} d \ln (q_{2t}) \end{bmatrix} \\ \tilde{x} &= x \otimes I_{2 \times 2} \quad \tilde{y} = \text{vec}(y) \\ \hat{\beta} &= (\tilde{x}' \tilde{x})^{-1} \tilde{x}' \tilde{y}\end{aligned}\quad (10)$$

Unequal Variances

- In our original formulation, we did not question the relative weighting of each equation. However, it is apparent that each equation will probably have a different variance.
- Thus, as a starting point, let us consider a heteroscedastic adjustment

$$\frac{\bar{w}_{1t} d \ln(q_{1t})}{\sqrt{\sigma_{11}}} = \theta_1 \frac{d \ln(Q_t)}{\sqrt{\sigma_{11}}} + \pi_{11} \frac{d \ln(p_{1t})}{\sqrt{\sigma_{11}}} + \pi_{12} \frac{d \ln(p_{2t})}{\sqrt{\sigma_{11}}} + \pi_{13} \frac{d \ln(p_{3t})}{\sqrt{\sigma_{11}}} + \frac{\epsilon_{1t}}{\sqrt{\sigma_{11}}}$$
$$\frac{\bar{w}_{2t} d \ln(q_{2t})}{\sqrt{\sigma_{22}}} = \theta_2 \frac{d \ln(Q_t)}{\sqrt{\sigma_{22}}} + \pi_{12} \frac{d \ln(p_{1t})}{\sqrt{\sigma_{22}}} + \pi_{22} \frac{d \ln(p_{2t})}{\sqrt{\sigma_{22}}} + \pi_{23} \frac{d \ln(p_{3t})}{\sqrt{\sigma_{22}}} + \frac{\epsilon_{2t}}{\sqrt{\sigma_{22}}} \quad (11)$$

Unequal Variances - Continued

- Thus, if we know σ_{11} and σ_{22} we can appropriately weight the equations.
 - The only bad part is that while it looks good, it really will not affect the estimates – we have just multiplied each equation by a constant.
- Following our discussion of $y_* = Py$ and $X_* = PX$ from the generalized least squares regressions we could do the same operation by

$$y_* = \begin{bmatrix} \bar{w}_{1t} d \ln(q_{1t}) \\ w_{2t} d \ln(q_{2t}) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 \\ 0 & \frac{1}{\sigma_{22}} \end{bmatrix} \quad (12)$$

$$X_* = \begin{bmatrix} d \ln(Q_t) & d \ln(p_{1t}) & d \ln(p_{2t}) & d \ln(p_{3t}) \\ d \ln(Q_t) & d \ln(p_{1t}) & d \ln(p_{2t}) & d \ln(p_{3t}) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 \\ 0 & \frac{1}{\sigma_{22}} \end{bmatrix}$$

Unequal Variances - Continued

- Estimating the residuals based on these results

$$\tilde{\epsilon} = \tilde{y} - \tilde{X}\beta_R \quad (13)$$

- Reshaping these residuals, we can then estimate the variance matrix

$$\hat{\Omega} = \hat{\epsilon}'\hat{\epsilon}/n \quad (14)$$

Unequal Variances - Continued

- Consider the regression in parts first X'_*y_*

$$\begin{bmatrix} d \ln(Q_t) & d \ln(p_{1t}) & d \ln(p_{2t}) & d \ln(p_{3t}) \\ d \ln(Q_t) & d \ln(p_{1t}) & d \ln(p_{2t}) & d \ln(p_{3t}) \end{bmatrix}' \begin{bmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 \\ 0 & \frac{1}{\sigma_{22}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\sigma_{11}}} & 0 \\ 0 & \frac{1}{\sigma_{22}} \end{bmatrix}' \\ \times \begin{bmatrix} \bar{w}_{1t} d \ln(q_{1t}) \\ w_{2t} d \ln(q_{2t}) \end{bmatrix} \quad (15)$$

- Given that $P'P = \Omega^{-1}$ we could write this expression as

$$\begin{bmatrix} d \ln(Q_t) & d \ln(p_{1t}) & d \ln(p_{2t}) & d \ln(p_{3t}) \\ d \ln(Q_t) & d \ln(p_{1t}) & d \ln(p_{2t}) & d \ln(p_{3t}) \end{bmatrix}' \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} \bar{w}_{1t} d \ln(q_{1t}) \\ w_{2t} d \ln(q_{2t}) \end{bmatrix} \quad (16)$$

Unequal Variances - Continued

- Expanding the regression formulation from Equation 10

$$\tilde{V} = I_{n \times n} \otimes \Omega^{-1} \Rightarrow \begin{bmatrix} \Omega^{-1} & 0 & \dots & 0 \\ 0 & \Omega^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Omega^{-1} \end{bmatrix} \quad (17)$$
$$\tilde{\beta}_{SUR} = (\tilde{X}' \tilde{V} \tilde{X})^{-1} \tilde{X}' \tilde{V} \tilde{y}$$

Unequal Variances - Continued

- Without loss Equation 17 can be used to consider "full variance matrices" or matrices with cross-equation covariance.

$$\Omega = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1k} & \sigma_{2k} & \cdots & \sigma_{kk} \end{bmatrix} \quad (18)$$

Restrictions in Seemingly Unrelated Regression

- Intuitively, we decided in Equation 10 that the estimated parameters would change – we have simply rescaled each equation.
- The restrictions (either exclusion restrictions so that one variable does not appear in one of the equations or cross-equation restrictions such as symmetry) make Seemingly Unrelated Regression estimates different than Ordinary Least Squares estimates.

Feasible Implementation of SUR

- We start with estimating the residuals using Ordinary Least Squares in the system

$$\beta_R = (\tilde{X}'\tilde{X})^{-1} (\tilde{X}'\tilde{y}) - (\tilde{X}'\tilde{X})^{-1} R' \left[R (\tilde{X}'\tilde{X})^{-1} R' \right]^{-1} (R (\tilde{X}'\tilde{X}) (\tilde{X}'\tilde{y}) - r) \quad (19)$$

- We can then use these Ordinary Least Squares Estimates to estimate the residuals

$$\tilde{\epsilon} = \tilde{y} - \tilde{X}\beta_R \quad (20)$$

- Unstacking the $\tilde{\epsilon}$ vector into a $n \times k$ matrix we can estimate the variance matrix as

$$\hat{\Omega} = \tilde{\epsilon}'\tilde{\epsilon}/n \quad (21)$$

Feasible Implementation of SUR – Continued

- Pulling everything together, we then reestimate adjusting for heteroscedasticity

$$\beta_{SUR} = \left(\tilde{X}' \hat{V}^{-1} \tilde{X} \right)^{-1} \left(\tilde{X}' \hat{V}^{-1} \tilde{y} \right) - \left(\tilde{X}' \hat{V}^{-1} \tilde{X} \right)^{-1} R' \left[R \left(\tilde{X}' \hat{V}^{-1} \tilde{X} \right)^{-1} R' \right]^{-1} \\ \times \left(R \left(\tilde{X}' \hat{V}^{-1} \tilde{X} \right) \left(\tilde{X}' \hat{V}^{-1} \tilde{y} \right) - r \right) \quad (22)$$

Six Good Rotterdam Model

Parameter	β_{OLS}	β_R	β_{SUR}	Relative Change	Stata Results
θ_1	0.2005	0.2034	0.2006	0.014	0.2006
θ_2	0.0260	0.0209	0.0243	0.147	0.0243
θ_3	0.3348	0.3364	0.3333	0.009	0.3333
θ_4	0.1997	0.1999	0.1988	0.005	0.1988
θ_5	0.0388	0.0445	0.0463	0.041	0.0463
π_{11}	-0.1140	-0.1104	-0.1019	0.080	-0.1019
π_{21}	0.0058	0.0027	0.0049	0.581	0.0049
π_{31}	0.0354	0.0228	0.0178	0.243	0.0178
π_{41}	0.0101	0.0307	0.0274	0.111	0.0274
π_{51}	0.0090	0.0249	0.0159	0.443	0.0159
π_{12}	-0.0086	0.0027	0.0049	0.581	0.0049
π_{22}	-0.0126	-0.0126	-0.0081	0.434	-0.0081
π_{32}	-0.0213	-0.0091	-0.0082	0.100	-0.0082
π_{42}	0.0081	0.0051	0.0011	1.315	0.0011
π_{52}	-0.0069	0.0075	0.0115	0.421	0.0115
π_{13}	0.0162	0.0228	0.0178	0.243	0.0178
π_{23}	-0.0065	-0.0091	-0.0082	0.100	-0.0082
π_{33}	-0.0509	-0.0459	-0.0488	0.061	-0.0488

Six Good Rotterdam Model - Continued

Parameter	β_{OLS}	β_R	β_{SUR}	Relative Change	Stata Results
π_{43}	0.0528	0.0533	0.0513	0.037	0.0513
π_{53}	-0.0001	-0.0089	-0.0064	0.326	-0.0064
π_{14}	0.0451	0.0307	0.0274	0.111	0.0274
π_{24}	0.0021	0.0051	0.0011	1.315	0.0011
π_{34}	0.0617	0.0533	0.0513	0.037	0.0513
π_{44}	-0.1046	-0.1096	-0.1026	0.066	-0.1026
π_{54}	0.0037	0.0075	0.0110	0.378	0.0110
π_{15}	0.0350	0.0249	0.0159	0.443	0.0159
π_{25}	0.0171	0.0075	0.0115	0.421	0.0115
π_{35}	-0.0107	-0.0089	-0.0064	0.326	-0.0064
π_{45}	0.0241	0.0075	0.0110	0.378	0.0110
π_{55}	-0.0293	-0.0330	-0.0291	0.125	-0.0291
π_{16}	0.0606	0.0653	0.0741	0.126	0.0741
π_{26}	0.0023	0.0164	0.0083	0.659	0.0083
π_{36}	-0.0962	-0.0944	-0.0881	0.070	-0.0881
π_{46}	0.0445	0.0509	0.0490	0.039	0.0490
π_{56}	0.0366	0.0169	0.0112	0.405	0.0112

Readings – Next Time

- * Bowden, R.J. and D.A. Turkington. 1984. *Instrumental Variables*. Cambridge University Press. (Chapters 1 and 2 pp. 1-65)
- * Cameron, A. C. and P. K. Trivedi. 2010. *Microeconometrics Using Stata Revised Edition*. Stata Press. (Chapter 6).
- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition (Chapter 8 Section 8.1 8.3 pp. 219 230)
- Radner, R. and F. Bobkoski. 1992. Some Recent Work of H. Theil on Estimation in Systems of Simultaneous Equations. In *Henri Theils Contributions to Economics and Econometrics: Volume I* (eds) Baldev, R. and J. Koerts. Kluwer Academic Publishers, 117-125.

Readings – Next Time

- * Theil, H.1992. Estimation and Simultaneous Correlation in Incomplete Equations Systems. In *Henri Theils Contributions to Economics and Econometrics: Volume I* (eds) Baldev,R. and J. Koerts. Kluwer Academic Publishers, 65-107.