

Testing Linear Restrictions - Restricted Least Squares: Lecture XVI

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Readings

- * Greene, W.H. 2012. Econometric Analysis Seventh Edition (Chapter 5 Section 5.1 5.2 pp. 108 112)
- * Hansen, B.E. 2011. Econometrics Draft Textbook Website: <http://www.ssc.wisc.edu/bhansen>. (Chapter 7).
- * Morgan, M.S. 1992. The History of Econometric Ideas. Cambridge University Press. (Chapter 6: The Evolution of Identification Questions pp. 162-190).

Restricted Least Squares

- Consider fitting the linear model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \epsilon \quad (1)$$

to the data presented in Table 5.

Regression Data for Restricted Least Squares

Observation	y	x ₁	x ₂	x ₃	x ₄
1	75.72173	4.93638	9.76352	4.39735	2.27485
2	45.11874	6.95106	3.11080	-1.96920	3.59838
3	51.61298	4.69639	4.17138	3.84384	2.73787
4	92.53986	10.22038	8.93246	1.73695	5.36207
5	118.74310	12.05240	12.22066	6.40735	4.92600
6	80.78596	10.42798	5.58383	1.61742	9.30154
7	43.79312	2.94557	5.16446	1.21681	4.75092
8	47.84554	3.54233	5.58659	2.18433	3.65499
9	63.02817	4.56528	6.52987	4.40254	5.36942
10	88.83397	11.47854	8.82219	0.70927	2.94652
11	104.06740	11.87840	8.53466	5.21573	8.91658
12	57.40342	7.99115	7.42219	-3.62246	-2.19067
13	76.62745	7.14806	7.39096	5.19569	3.00548
14	109.96540	10.34953	9.82083	7.82591	7.09768
15	72.66822	7.74594	4.79418	5.39538	6.29685
16	68.22719	4.10721	8.51792	4.00252	3.88681
17	122.50920	12.77741	11.57631	6.85352	7.63219
18	70.71453	9.69691	6.54209	0.53160	0.79405
19	70.00971	6.46460	6.62652	4.31049	5.03634
20	75.82481	6.31186	8.49487	3.38461	5.53753
21	38.82780	3.04641	2.99413	2.69198	6.26460
22	79.15832	8.85780	7.29142	3.33994	2.86917
23	62.29580	5.82182	6.16096	4.18066	1.73678
24	80.63698	4.97058	9.83663	6.71842	3.47608
25	77.32687	5.90209	8.56241	5.42130	4.70082
26	23.34500	1.57363	2.82311	0.95729	0.69178
27	81.54044	9.25334	6.43342	5.02273	3.84773
28	67.16680	10.77622	5.52171	-0.87349	1.17348

Unrestricted Ordinary Least Squares

- Solving for the least squares estimates

$$\hat{\beta} = (X'X)^{-1} (X'y) = \begin{pmatrix} 4.7238 \\ 4.0727 \\ 3.9631 \\ 2.0185 \\ 0.9071 \end{pmatrix} \quad (2)$$

Estimating the variance matrix

$$\hat{s}^2 = \frac{y'y - (y'X) (X'X)^{-1} (X'y)}{30 - 5} = 1.2858 \quad (3)$$

$$V(\hat{\beta}) = \hat{s}^2 (X'X)^{-1}$$

Unrestricted Ordinary Least Squares - Continued

- Estimated Variance Matrix

$$V(\hat{\beta}) = \begin{bmatrix} 0.5037 & -0.0111 & -0.0460 & 0.0252 & -0.0285 \\ -0.0111 & 0.0079 & -0.0068 & 0.0044 & -0.0033 \\ -0.0460 & -0.0068 & -0.0164 & -0.0104 & 0.0047 \\ 0.0252 & 0.0044 & -0.0104 & 0.0141 & -0.0070 \\ -0.0285 & -0.0033 & 0.0047 & -0.0070 & 0.0104 \end{bmatrix} \quad (4)$$

Restrictions on Least Squares

- Next, consider the hypothesis that $\beta_1 = \beta_2$ (which seems plausible given the results above).
 - As a starting point, consider the least squares estimator as a constrained minimization problem

$$\begin{aligned} L(\beta) &= (y - X\beta)'(y - X\beta) + 2\lambda'(r - R\beta) \\ &= (y' - \beta'X')(y - X\beta) + 2\lambda'(r - R\beta) \end{aligned} \quad (5)$$

$$\nabla_{\beta'} L(\beta) = X'(y - X\beta) + X'(y - X\beta) - 2R'\lambda = 0$$

- The second term in the gradient vector depends on the vector derivative

$$\nabla_{\beta'}(X\beta) = \nabla_{\beta'}(\beta'X')' = (\nabla_{\beta'}(\beta'X'))' = (X')' = X \quad (6)$$

Restrictions on Least Squares - Continued

- Continued
 - Solving for the first-order condition for β

$$\begin{aligned}
 X'(y - X\beta) + X'(y - X\beta) - 2R'\lambda &= 0 \\
 2(X'y) - 2(X'X)\beta - 2R'\lambda &= 0 \\
 (X'X)\beta &= (X'y) - R'\lambda \\
 \beta &= (X'X)^{-1}(X'y) - (X'X)^{-1}R'\lambda
 \end{aligned} \tag{7}$$

- Taking the gradient of the Lagrange formulation with respect to λ yields

$$\nabla_{\lambda'} L(\beta) = r - R\beta = 0 \tag{8}$$

Restrictions on Least Squares - Continued

- Continued

- Substituting the solution of β into the first-order condition with the respect to the Lagrange multiplier

$$\begin{aligned} r - R \left[(X'X)^{-1} (X'y) - (X'X)^{-1} R' \lambda \right] &= 0 \\ R - R (X'X)^{-1} (X'y) + R (X'X)^{-1} R' \lambda &= 0 \\ R (X'X)^{-1} R' \lambda &= R (X'X)^{-1} (X'y) - r \\ \lambda &= \left[R (X'X)^{-1} R' \right]^{-1} \left(R (X'X)^{-1} (X'y) - r \right) \end{aligned} \quad (9)$$

- Note that substituting $\beta = (X'X)^{-1} (X'y)$ into this expression yields

$$\lambda = \left[R (X'X)^{-1} R' \right]^{-1} (R\beta - r) \quad (10)$$

Restrictions on Least Squares - Continued

- Continued
 - Substituting this result for λ back into the first-order conditions with respect to β yields

$$\begin{aligned}\beta_R &= (X'X)^{-1} (X'y) - (X'X)^{-1} R' \left[R' (X'X)^{-1} R' \right]^{-1} (R(X'X)(X'y) - r) \\ &= \beta - (X'X)^{-1} R' [R(X'X)R']^{-1} (R\beta - r)\end{aligned}\quad (11)$$

Restrictions on Least Squares - Continued

- Given this derivation $\beta_1 = \beta_2$ implies

$$R\beta - r = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = 0 \quad (12)$$

Yielding

$$\beta_R = \begin{bmatrix} 4.6122 \\ 4.0302 \\ 4.0302 \\ 1.9752 \\ 0.9303 \end{bmatrix} \quad (13)$$

Variance of Restricted Least Squares

- Starting by deriving $\beta - E[\beta]$ based on the previous results

$$\beta = (X'X)^{-1} (X'X\beta + X'\epsilon) - (X'X)^{-1} R' \left[R (X'X)^{-1} R' \right]^{-1} \\ \times \left(R (X'X)^{-1} X'\beta + R (X'X)^{-1} X'\epsilon - r \right)$$

$$E[\beta] = (X'X)^{-1} (X'X) \beta - (X'X)^{-1} R' \left[R (X'X)^{-1} R' \right]^{-1} \\ \times \left(R (X'X)^{-1} X'\beta - r \right) \quad (14)$$

$$\beta - E[\beta] = (X'X)^{-1} X'\epsilon - (X'X)^{-1} R' \left[R (X'X)^{-1} R' \right]^{-1} R \\ \times (X'X)^{-1} X'\epsilon$$

Variance of Restricted Least Squares - Continued

- Computing $(\beta - E[\beta])(\beta - E[\beta])'$ based on this result

$$\begin{aligned}
 (\beta - E[\beta])(\beta - E[\beta])' &= (X'X)^{-1} X' \epsilon \epsilon' X (X'X)^{-1} - \\
 &- (X'X)^{-1} X' \epsilon \epsilon' X (X'X)^{-1} R' [R (X'X) R']^{-1} R (X'X)^{-1} \\
 &- (X'X)^{-1} R' [R (X'X)^{-1} R']^{-1} R (X'X)^{-1} X' \epsilon \epsilon' X (X'X)^{-1} \\
 &+ (X'X)^{-1} R' [R (X'X)^{-1} R']^{-1} R (X'X)^{-1} X' \epsilon \epsilon' \\
 &\quad \times X (X'X)^{-1} R' [R (X'X)^{-1} R']^{-1} R (X'X)^{-1}
 \end{aligned}
 \tag{15}$$

Variance of Restricted Least Squares - Continued

- Taking the Expectation of both sides and noting that $E[\epsilon\epsilon'] = \sigma^2 I$

$$\begin{aligned}
 V(\beta) &= E[(\beta - E[\beta])(\beta - E[\beta])'] \\
 &= \sigma^2 \left[(X'X)^{-1} - (X'X)^{-1} R' [R(X'X)^{-1} R']^{-1} R (X'X)^{-1} \right]
 \end{aligned}
 \tag{16}$$

F Test of Restrictions

- Starting with the derivation of the error under the restriction

$$\epsilon_R = y - X\beta_R \quad (17)$$

$$= y - X\beta - X(\beta_R - \beta)$$

- Computing the variance under the restriction

$$\begin{aligned} \epsilon_R' \epsilon_R &= (\epsilon - X(\beta_R - \beta))' (\epsilon - X(\beta_R - \beta)) \\ &= \epsilon' \epsilon - \epsilon' X(\beta_R - \beta) - (\beta_R - \beta)' X' \epsilon + (\beta_R - \beta)' X' X (\beta_R - \beta) \end{aligned} \quad (18)$$

F Test of Restrictions - Continued

- Taking the expectation of both sides (with $E[\epsilon] = 0$)

$$\begin{aligned} E(\epsilon'_R \epsilon_R) &= E(\epsilon' \epsilon) + (\beta_R - \beta)' X' X (\beta_R - \beta) \\ \Rightarrow E(\epsilon'_R \epsilon_R) - E(\epsilon' \epsilon) &= (\beta_R - \beta)' X' X (\beta_R - \beta) \end{aligned} \quad (19)$$

- From our forgoing discussion

$$\beta_R - \beta = (X'X)^{-1} R' \left[R (X'X)^{-1} R' \right]^{-1} (r - R\beta) \quad (20)$$

F Test of Restrictions - Continued

- Substituting this result back into the previous equation yields

$$\begin{aligned}
 E(\epsilon'_R \epsilon_R) - E(\epsilon' \epsilon) &= (r - R\beta)' \left[R (X'X)^{-1} R' \right]^{-1} \\
 R (X'X)^{-1} X'X (X'X)^{-1} R' \left[R (X'X)^{-1} R' \right]^{-1} (r - R\beta) &\quad (21) \\
 &= (r - R\beta)' \left[R (X'X)^{-1} R' \right]^{-1} (r - R\beta)
 \end{aligned}$$

F Test of Restrictions - Continued

- Therefore, the test for these linear restrictions becomes

$$F(q, n - k) = \frac{(\epsilon'_R \epsilon_R - \epsilon' \epsilon) / q}{\epsilon' \epsilon / (n - k)} \quad (22)$$

$$= \frac{(r - R\beta)' \left[R (X'X)^{-1} R' \right]^{-1} (r - R\beta) / q}{\epsilon' \epsilon / (n - k)}$$

where there are q restrictions, n observations, and k independent variables.

F Tests of Restrictions - Continued

- Building on the definition of the variance matrix Greene (pp. 118-119) demonstrates

$$F = \frac{(r - R\beta)' \left[R (X'X)^{-1} R' \right]^{-1} (r - R\beta) / q}{\epsilon' \epsilon / (n - k)} \quad (23)$$

$$= \frac{(r - R\beta)' \left[\sigma^2 R (X'X)^{-1} R' \right]^{-1} (r - R\beta) / q}{[(n - k) s^2 / \sigma^2] / (n - k)}$$

- To emphasize the χ^2 / χ^2 formulation.

F Tests of Restrictions - Continued

- But it does allow for the expression

$$F[q, n - k] = \frac{(r - R\beta)' \left[R \left(s^2 (X'X)^{-1} \right) R' \right]^{-1} (r - R\beta)}{q} \quad (24)$$

Readings - Next Time

- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition (Chapter 5 Section 5.3 5.7 pp. 114 133)
- * Engle, R.F. 2004. Chapter 13: Wald, Likelihood Ratio, and Lagrange Multiplier Tests in Econometrics. *Handbook of Econometrics* Volume 2, 775-826.
(<http://www.sciencedirect.com/science/article/pii/S15734412840200>)