

Testing Ordinary Least Squares Results in the Small and Large - Asymptotic Normality: Lecture XV

Charles B. Moss ¹

¹University of Florida

February 6, 2012

- 1 Readings
- 2 Asymptotic Distributions
 - Fourier Approximations
 - Moment Generating Functions
 - Characteristic Function
- 3 Central Limit Theory
- 4 Readings - Next Time

Readings

- Folland, G.B. 2000. *Fourier Analysis and its Applications*. American Mathematical Society.
- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition. (Chapter 4 Sections 4.4-4.3 pp. 63 - 87)
- * Hansen, B.E. 2011. *Econometrics* Draft Textbook Website: <http://www.ssc.wisc.edu/bhansen>. (Chapter 6).
- * Morgan, M.S. 1992. *The History of Econometric Ideas*. Cambridge University Press. (Chapter 4: Tinbergen and Macrodynamic Models pp. 101-130).
- Theil, H. 1971. *Principles of Econometrics* John Wiley & Sons. (Chapter 8)
- * White, H. 2000. *Asymptotic Theory for Econometricians* Revised Edition. Academic Press.

Asymptotic Distributions

- In Lecture XIV we were primarily interested in determining that ordinary least squares estimates existed in the limit.
- In this lecture we are interested in developing the distribution of those estimates.
- In particular, we are interested in the cases where the asymptotic limit of those distributions is normal.
- Two cases:
 - 1 Applying the central limit theorem to ordinary least squares estimates implies normality because these estimates are consistent.
 - 2 Applying the central limit theorem to maximum likelihood estimates using Cramer-Rao.

Fourier Approximations

- Starting our discussion with the notion of approximating an unknown function $g(x)$.
 - The standard Taylor-Series approach can be written as

$$\begin{aligned} \tilde{g}(x) \approx & f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \\ & \frac{1}{6}f'''(x_0)(x - x_0)^3 + \frac{1}{24}f''''(x_0)(x - x_0)^4 \end{aligned} \quad (1)$$

- We can then approximate the original function as

$$g(x) = \alpha_0 + \alpha_1x + \alpha_2x^2 + \alpha_3x^3 + \alpha_4x^4 \quad (2)$$

Fourier Approximation - Continued

- From trigonometry we have the Euler's formula (Folland p. 18)

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \sin(\theta) = \frac{e^{-i\theta} - e^{-i\theta}}{2} \quad (3)$$

$$e^{ix} = \cos(x) + i \sin(x)$$

- What we want to do is to come up with an expansion (like the Taylor series expansion) for a function so that

$$f(\theta) = \frac{1}{2}a_0 + \sum_{l=1}^{\infty} (a_l \cos(l\theta) + b_l \sin(l\theta)) \quad (4)$$

Fourier Approximation - Continued

- Or, as in the case of the Taylor series expansion

$$f(x) = \alpha_0 + \sum_{i=1}^N (a_i \cos(\pi_i x) + b_i \sin(\pi_i x)) \quad (5)$$

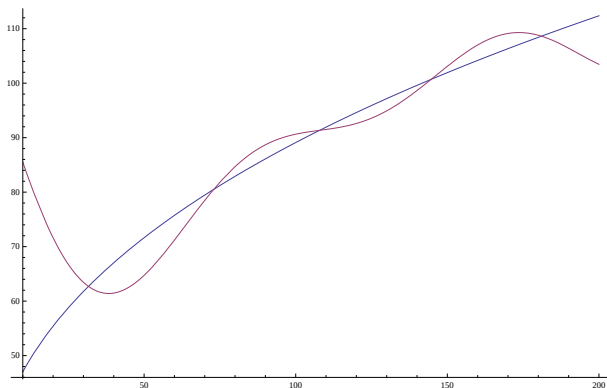
where the π_i are dominant periodicities from the infinite form of the Fourier.

- Consider the Leontief production function

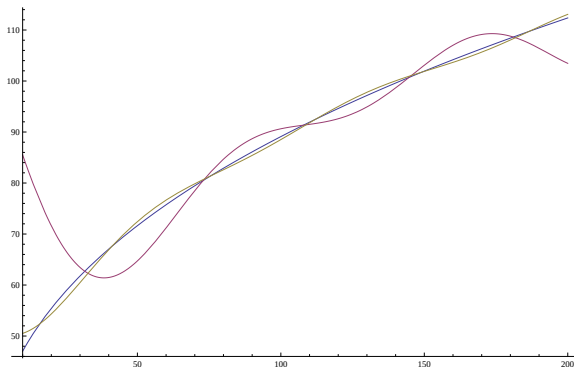
$$f(n, p) = -22.3029 - 0.0491n - 0.5367p + 6.5213n^{1/2} + 10.8001p^{1/2} + 0.04183n^{1/2}p^{1/2} \quad (6)$$

- Fixing $p = 45$ we attempt to approximate the function first using a Fourier and then with a Taylor expansion.

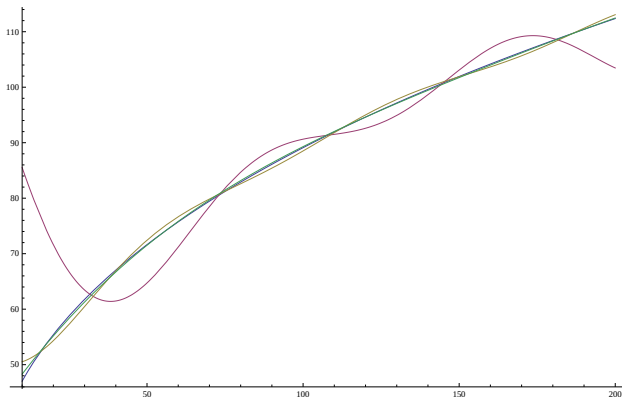
Approximating with Fourier $n = 3$



Approximating with Fourier $n = 4$



Approximating with Fourier and Taylor



Moment Generating Functions

- **Definition 3.26**[Moss p. 53] Let X be a random variable with cumulative distribution function $F(X)$. The moment generating function (mgf) of X (or $F(X)$), denoted $M_X(t)$, is

$$M_X(t) = E[e^{tx}] \quad (7)$$

provided that the expectation exists for t in some neighborhood around 0.

- **Theorem 3.27**[Moss p.53] If X has mgf $M_X(t)$, then

$$E[X^n] = M_X^{(n)}(0) \quad (8)$$

where we define

$$M_X^{(n)}(0) = \left. \frac{d^n M_X(t)}{dt^n} \right|_{t \rightarrow 0} \quad (9)$$

Moment Generating Function - Continued

- Taking the Taylor series expansion around $t \rightarrow 0$

$$\begin{aligned} M_X(t)|_{x_0 \rightarrow 0} &= E[e^{tx}] \\ &= E\left[e^0 + te^{t \times 0}(x - 0) + \frac{1}{2}t^2e^{t \times 0}(x - 0)^2 + \dots\right] \quad (10) \\ &= 1 + E[x]t + E[x^2]\frac{t^2}{2} + E[x^3]\frac{t^3}{6} \end{aligned}$$

Thus, for any moment n

$$M_X^{(n)}(t) = \frac{d^n}{dt^n} M_X(t) = E[x^n] + tE[x^{n+1}] + t^2E[x^{n+2}] + \dots \quad (11)$$

- Two random variables with the same moment generating function have the same distribution.

Characteristic Function

- **Definition 5.26**[Moss p.83] Let Z be a $k \times 1$ random vector with distribution function F . The characteristic function of F is defined as

$$f(\lambda) = E[\exp(i\lambda'Z)] \quad (12)$$

where $i^2 = -1$ and λ is a $k \times 1$ real vector.

- For a normal distribution $Z \sim N(\mu, \sigma^2)$

$$f(\lambda) = \exp\left(i\lambda\mu - \frac{\lambda^2\sigma^2}{2}\right) \quad (13)$$

Characteristic Function - Continued

- Taking each $f(\lambda)$ to the characteristic function for $Z_t - \mu$ and letting $f_n(\lambda)$ be the characteristic function for the random variable

$$\frac{\sqrt{n}(\bar{Z}_n - \bar{\mu}_n)}{\bar{\sigma}_n} = \left(\frac{1}{n}\right)^{1/2} \sum_{i=1}^n \left(\frac{Z_t - \mu}{\sigma}\right). \quad (14)$$

- By this structure

$$f_n(\lambda) = f\left(\frac{\lambda}{\sigma\sqrt{n}}\right) \quad (15)$$

$$\ln(f_n(\lambda)) = n \ln\left(f\left(\frac{\lambda}{\sigma\sqrt{n}}\right)\right)$$

Characteristic Function - Continued

- Taking the second-order Taylor series expansion around an arbitrary distribution $f(\lambda)$ around $\lambda = 0$

$$f(\lambda) = 1 - \frac{\sigma^2 \lambda^2}{2} + o(\lambda^2) \quad (16)$$

Thus

$$\ln(f_n(\lambda)) = n \ln \left[1 - \frac{\lambda^2}{2n} + o\left(\frac{\lambda^2}{n}\right) \right] \rightarrow \frac{\lambda^2}{2} \text{ as } n \rightarrow \infty \quad (17)$$

Central Limit Theory

- Proposition 5.0** Given restrictions on the moments, dependence, and heterogeneity of a scalar sequence $\{Z_t\}$, $(\bar{Z}_n - \bar{\mu}_n) / (\bar{\sigma}_n / \sqrt{n}) = \sqrt{n} (\bar{Z}_n - \bar{\mu}_n) / \bar{\sigma}_n \overset{A}{\sim} N(0, 1)$ where $\bar{\mu}_n \equiv E(\bar{Z}_n)$ and $\bar{\sigma}^2/n(\bar{Z}_n)$.
- Theorem 5.2 (Lindeberg Lévy)** Let $\{Z_t\}$ be a sequence of i.i.d. random scalars, with $\mu \equiv E[Z_t]$ and $\sigma^2 \equiv \text{var}[Z_t] < \infty$. If $\sigma^2 \neq 0$, then

$$\begin{aligned} \sqrt{n} (\bar{Z}_n - \bar{\mu}_n) / \sigma_n &= \sqrt{n} (\bar{Z}_n - \mu) / \sigma \\ &= n^{-1/2} \sum_{t=1}^n (Z_t - \mu) / \sigma \overset{A}{\sim} N(0, 1) \end{aligned} \tag{18}$$

Central Limit Theory

• Theorem 5.3 Given

- i. $Y_t = X_t' \beta_0 + \epsilon_t$, $t = 1, 2, \dots$, $\beta_0 \in R^k$;
- ii. $\{ (X_t' \epsilon_t) \}$ is an i.i.d. sequence;
- iii. Expectations
 - a. $E[X_t' \epsilon_t] = 0$;
 - b. $E|X_{thi} \epsilon_{th}|^2 < \infty$, $h = 1, \dots, p$, $i = 1, \dots, k$;
 - c. $V_n \equiv \text{var} \left(n^{-1/2} X' \epsilon \right) = V$ is positive definite;
- iv. Boundedness
 - a. $E|X_{thi}^2| < \infty$, $h = 1, \dots, p$, $i = 1, \dots, k$;
 - b. $M \equiv E(X_t' X_t)$ is positive definite.

- Then $D^{-1/2} \sqrt{n} \left(\hat{\beta}_n - \beta_p \right) \overset{A}{\sim} N(0, I)$, where $D \equiv M^{-1} V M^{-1}$.

Suppose in addition that

- v. there exists $\hat{V}_n$ symmetric and positive semidefinite such that

$$\hat{V}_n - V \xrightarrow{P} 0.$$

- Then $\hat{D}_n - D \xrightarrow{P} 0$, where $\hat{D}_n = (X'X/n)^{-1} \hat{V}_n (X'X/n)^{-1}$.

Readings - Next Time

- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition (Chapter 5 Section 5.1 5.2 pp. 108 112)
- * Hansen, B.E. 2011. *Econometrics* Draft Textbook Website: <http://www.ssc.wisc.edu/bhansen>. (Chapter 7).
- * Morgan, M.S. 1992. *The History of Econometric Ideas*. Cambridge University Press. (Chapter 6: The Evolution of Identification Questions pp. 162-190).