

Testing Ordinary Least Squares Results in the Small and Large - Asymptotic Results: Lecture XIV

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Readings

- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition. (Chapter 4 Sections 4.4-4.3 pp. 63 - 87)
- * Hansen, B.E. 2011. *Econometrics* Draft Textbook Website: <http://www.ssc.wisc.edu/bhansen>. (Chapter 6).
- * Morgan, M.S. 1992. *The History of Econometric Ideas*. Cambridge University Press. (Chapter 4: Tinbergen and Macrodynamic Models pp. 101-130).
- Theil, H. 1971. *Principles of Econometrics* John Wiley & Sons. (Chapter 8)

Distribution of Regression Coefficients

- Lectures XIII, XIV, and XV are largely about the distributions of the estimated parameters from Ordinary Least Squares
- Lecture XIII covered the “standard case” or the “simple case” where the residuals from the regression are assumed to be independent and identically distributed normal.
- The rest of this course is really related to failures of this assumption.
- These failures may affect the nature of the estimator.
 - Under heteroscedasticity the estimator is still unbiased, but the estimates are inefficient.
 - Under the correlated excluded variable problem, the estimates are no longer unbiased.

Distribution of Regression Coefficients - Continued

- Apart from affecting the estimator, these changes in the properties of the errors affect the statistical properties of the estimates (i.e., the ordinary least squares estimates may no longer be normal).
- As an example, consider a linear sample drawn from the distribution

$$f(z|\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta, X) = \frac{1}{\Gamma(\alpha(X)) \beta^{\alpha(X)}} z^{(\alpha(X)-1)} \exp(-z/\beta)$$

$$\ni: \alpha(X) = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$$

(1)

Distribution of Regression Coefficients - Continued

- In this regression the random errors are gamma such that

$$\begin{aligned} E[z] &= (\alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3) \beta \\ V[z] &= (\alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3) \beta^2 \end{aligned} \quad (2)$$

- The coefficients and statistical properties of those coefficients can be estimated using maximum likelihood.
- However, on question is – Could we estimate the parameters using ordinary least squares?

Fixed in Sample or Random

- As often is the case, the mechanics used to develop one problem also lend themselves to another difficulty.
- In this case, the asymptotic theory used to develop the test under non-normality is also useful for examining the effect of the randomness of X .
- In our proof of Gauss-Markov, we constructed the variance matrix for estimated coefficients as

$$\begin{aligned} V(\hat{\beta}) &= E \left[(X'X)^{-1} X' \epsilon \epsilon' X (X'X)^{-1} \right] \\ &= (X'X)^{-1} X' \sigma^2 I X (X'X)^{-1} \end{aligned} \quad (3)$$

- Basically, we are dealing with two types of assumptions:
 - Since X is fixed $E(X'X) = X'X$ and $V(X'X) = 0$, and
 - Since the expected error is equal to zero $E(X'\epsilon) = 0$.

Fixed in Sample or Random - Continued

- In experimental sciences it is possible to control many of the factors of the experiment (i.e., to fix the values of X).
- Given this ability there are even textbooks on optimally selecting the levels of control to maximize the information in the sample (i.e., to reduce the cost of the experiment).
- Typically in economics we cannot control the experiment.
 - In the KLEM data no one selected the values logarithmic changes in prices.
 - Typically, the President, Senate, and House of Representatives will not allow us to crash the economy on purpose to test an experiment.
- In fact, the lack of control leads to some other problems such as correlation with excluded factors.

Consistency

- Up to this point we have been interested in “unbiasedness” so that $E[\hat{\beta}] = \beta$.
- Now we make a subtitle shift to consistency

$$\hat{\beta} \xrightarrow{\text{a. s.}} \beta \quad (4)$$

read as $\hat{\beta}$ converges **almost surely** to β .

- A simple example is the case of the Bayesian estimator whose value is biased by the prior (initial guess at the parameters).
 - This bias has a large effect in small sample and its value will never completely go away.
 - However, as the sample becomes very large (in most cases), the effect of the bias can be made arbitrarily small.

Limits

- Most of this material is from Chapter 2 of White.
- What we want to develop is an intuition behind this convergence. A starting point is the notion of limits

Definition 2.1 *Let $\{b_n\}$ be a sequence of real numbers. If there exists a real number b and if for every real $\delta > 0$ there exists an integer $N(\delta)$ such that for all $n \geq N(\delta)$, $|b_n - b| < \delta$, then b is the **limit** of the sequence $\{b_n\}$.*

- Also in White (Definition 2.3) we see that the concept of limits extends to functions so that as $b_n \rightarrow b$ then $g(b_n) \rightarrow g(b)$ which opens the door to most of our statistical measures.

Measure Theory and Limits

- Most students of econometrics have been introduced to probability theory taking a short-cut.
 - In most of our discussion, we talk about X being a random variable with a given distribution.
 - In fact, a more complete characterization is that the random variable is defined as

$$X_n : \Omega \rightarrow R \quad (5)$$

where Ω is a space of possible random events and R is the set of real numbers.

- $X_n(\omega)$ is a realization of that random event.
- Based on measure theory under certain characteristics we can assign these events a probability measure $P\{\omega : X_n(\omega) \leq a\}$

.

Measure Theory and Limits - Continued

- An outcome of our estimator can then be written as a measure based on a particular sample drawn from the event space

$$\hat{\beta}_n(\omega) = \frac{1}{n} \sum_{t=1}^N Z_t(\omega) \quad (6)$$

Most of the time we ignore the event arguments.

- However, the event space arguments are useful in defining convergence.

Definition 2.9 Let $\{b_n(\cdot)\}$ be a sequence of real-valued random variables. We say that $b_n(\cdot)$ **converges almost surely** to b , written $b_n(\cdot) \xrightarrow{\text{a.s.}} b$ if there exists a real number b such that $P\{\omega : b_n(\omega) \rightarrow b\} = 1$.

Convergence and Ordinary Least Squares

- To develop the ordinary least squares estimator in this context we introduce Proposition 2.11

Proposition 2.11 *Given $g : R^k \rightarrow R^l$ ($k, l \in N$) and any sequence of random $k \times 1$ vectors $\{b_n\}$ such that $b_n \xrightarrow{\text{a. s.}} b$, where b is $k \times 1$, if g is continuous at b , then $g(b_n) \xrightarrow{\text{a. s.}} g(b)$.*

Convergence and Ordinary Least Squares - Continued

Theorem 2.12 *Suppose*

- $Y_t = X_t' \beta_0 + \epsilon_t, t = 1, 2, \dots, \beta_0 \in R^k ;$
- $X' \epsilon / n \xrightarrow{\text{a. s.}} 0 ;$
- $X' X / n \xrightarrow{\text{a. s.}} M, \text{ finite and positive definite.}$

Then $\hat{\beta}_0$ exists for all n sufficiently large a.s., and $\hat{\beta}_n \xrightarrow{\text{a. s.}} \beta_0 .$

- Ordinary least squares are consistent.