

Applying Ordinary Least Squares on the Computer

- Stata: Lecture XII

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1 Readings

2 System Maximum Likelihood

- Defining the Likelihood Function
- Derivative of the Likelihood Function
- Symmetry Restricted Maximum Likelihood

Readings

- * Cameron, A. C. and P. K. Trivedi. 2010. *Microeconometrics Using Stata* Revised Edition. Stata Press. (Chapters 3).
- * Morgan, M.S. 1992. *The History of Econometric Ideas*. Cambridge University Press. (Chapter 2: Measuring and Representing Business Cycles pp. 40 - 72).

System Maximum Likelihood

- Building on the Seemingly Unrelated Regression specification from the last lecture, we consider the multivariate maximum likelihood formulation.

$$\begin{aligned} L(\beta|x) &= -\frac{T}{2} \ln |\Omega| - \sum_{t=1}^T (y_t - \beta X'_t)' \Omega^{-1} (y_t - \beta X'_t) \\ &\quad \ni: y_t \in M_{k \times 1}, \beta \in M_{k \times m}, X'_t \in M_{m \times 1} \\ &\Rightarrow \Omega_{MLE}(\beta) = \sum_{t=1}^T (y_t - \beta X'_t) (y_t - \beta X'_t)' / T \\ &\Rightarrow L(\beta|x) = -\frac{T}{2} \ln \left| \sum_{t=1}^T (y_t - \beta X'_t) (y_t - \beta X'_t)' / T \right| - Tk \end{aligned} \tag{1}$$

System Maximum Likelihood - Continued

- To implement this formulation we rewrite the system as

$$\epsilon_t = \begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{bmatrix} - \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{13} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{14} & \beta_{23} & \beta_{34} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \\ x_{4t} \end{bmatrix} \quad (2)$$

Derivative of the Likelihood Function

- Remember the system normal likelihood function

$$L = -\frac{T}{2} \ln |\Omega| - \frac{1}{2} \sum_{t=1}^T \epsilon_t(\beta)' \Omega^{-1} \epsilon_t(\beta) \quad (3)$$

taking the derivative with respect to beta and applying the transpose given that $\nabla_{\beta} f(\beta)' = (\nabla_{\beta} f(\beta))'$, therefore
 $\nabla_{\beta} \beta' X' y \Rightarrow \nabla_{\beta} \beta' Z \Rightarrow (\nabla_{\beta} Z' \beta) = Z'.$

$$\nabla_{\beta} L = - \sum_{t=1}^T \epsilon_t(\beta)' \Omega^{-1} \nabla_{\beta} \epsilon_t(\beta) \quad (4)$$

Derivative of the Likelihood Function - Continued

- In the bivariate regression

$$\begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} = \begin{bmatrix} y_{1t} - \beta_{10} - \beta_{11}x_{1t} - \beta_{12}x_{2t} \\ y_{2t} - \beta_{20} - \beta_{21}x_{1t} - \beta_{22}x_{2t} \end{bmatrix} \quad (5)$$

Taking the matrix derivative of $\epsilon_t(\beta)$ implies

$$\nabla_{\beta} \epsilon_t(\beta) = \begin{bmatrix} -1 & -x_{1t} & -x_{2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -x_{1t} & -x_{2t} \end{bmatrix} \quad (6)$$

Derivative of the Likelihood Function - Continued

- Think about the gradient

$$\begin{aligned}\nabla_{\beta} L &= - \sum_{t=1}^T \epsilon_t(\beta)' \Omega^{-1} \nabla_{\beta} \epsilon_t(\beta) \\ \Rightarrow \nabla_{\beta} L &= \sum_{t=1}^T z_t \\ \ni: z_t &= \epsilon_t(\beta)' \Omega^{-1} \begin{bmatrix} -1 & -x_{1t} & -x_{2t} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -x_{1t} & -x_{2t} \end{bmatrix} \end{aligned} \quad (7)$$

The result will be a 1×6 matrix.

Symmetry Restricted Maximum Likelihood

- Imposing the restriction that $\beta_{12} = \beta_{21}$ the errors become

$$\begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} = \begin{bmatrix} y_{1t} - \beta_{10} - \beta_{11}x_{1t} - \beta_{12}x_{2t} \\ y_{2t} - \beta_{20} - \beta_{12}x_{1t} - \beta_{22}x_{2t} \end{bmatrix} \quad (8)$$

Taking the matrix derivative of $\epsilon_t(\beta)$ implies

$$\nabla_{\beta} \epsilon_t(\beta) = \begin{bmatrix} -1 & -x_{1t} & -x_{2t} & 0 & 0 \\ 0 & 0 & -x_{1t} & -1 & -x_{2t} \end{bmatrix} \quad (9)$$

$$\hat{\beta}' = [\beta_{10} \quad \beta_{11} \quad \beta_{12} \quad \beta_{20} \quad \beta_{22}]$$

Readings - Next Time

- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition. (Chapter 5)
- * Morgan, M.S. 1992. *The History of Econometric Ideas*. Cambridge University Press. (Chapter 3: Random Shocks Enter the Business Cycle pp. 73 - 100).
- * Theil, H. 1971. *Principles of Econometrics* John Wiley & Sons. (Chapter 3 Section 3.7)