

Linear Algebra and Matrix Methods - Part II: Lecture IV

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Readings

- Anderson, T.W. 1984. *An Introduction to Multivariate Statistical Analysis* Second Edition. John Wiley & Sons. (Chapter 13 pp. 521-549 and A.2 pp. 587-590).
- * Greene, W.H. 2012. *Econometric Analysis* Seventh Edition. Prentice Hall. (Appendix A: Matrix Algebra pp. 973-1014).
- * Dhrymes, P.J. 2000. *Mathematics for Econometrics* Third Edition. Springer-Verlag. (Chapter 2 [Section 2.8 - 2.12]).
- * Popper, K. 2010. *The Logic of Scientific Discovery* Routledge Classics [Chapter 4: Falsifiability, 57-73].
- * Theil, H. 1971. *Principles of Econometrics* John Wiley & Sons. (Chapter 1 [Sections 1.0 - 1.5]).
- * Theil, H. 1983. Chapter 1: Linear Algebra and Matrix Methods in Econometrics. In *Handbook of Econometrics*:

Kronecker Products

- As a starting point, I would like for you to consider two related equations

$$\begin{aligned} y_{1t} &= \alpha_{01} + \alpha_{11}x_{1t} + \alpha_{21}x_{2t} + \epsilon_{1t} \\ y_{2t} &= \alpha_{02} + \alpha_{12}x_{1t} + \alpha_{22}x_{2t} + \epsilon_{2t} \end{aligned} \quad (1)$$

- We could do this analysis with two separate regressions, but there may be reasons that we will discuss later to do them at the same time.
- Consider a specification for time $t = 1$

$$\begin{aligned} y_{11} &= \alpha_{01}1 + \alpha_{02}0 + \alpha_{11}x_{11} + \alpha_{12}0 + \alpha_{21}x_{21} + \alpha_{22}0 + \epsilon_{11} \\ y_{21} &= \alpha_{01}0 + \alpha_{02}1 + \alpha_{11}0 + \alpha_{12}x_{11} + \alpha_{21}0 + \alpha_{22}x_{21} + \epsilon_{21} \end{aligned} \quad (2)$$

Kronecker Products - Continued

- Rewriting this expression slightly

$$\begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_{11} & 0 & x_{21} & 0 \\ 0 & 1 & 0 & x_{11} & 0 & x_{21} \end{bmatrix} \begin{bmatrix} \alpha_{01} \\ \alpha_{02} \\ \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \end{bmatrix} + \begin{bmatrix} \epsilon_{11} \\ \epsilon_{21} \end{bmatrix} \quad (3)$$

- Kronecker products allow us to construct with new X matrix with little nomenclature.

Kronecker Products - Continued

- As described in Dhrymes (2000, p.49)

$$A \otimes B = \begin{bmatrix} a_{11}B & A_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{bmatrix} \quad (4)$$

- In this case

$$X_t = \begin{bmatrix} 1 & x_{1t} & x_{2t} \end{bmatrix}, \quad I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (5)$$

Kronecker Products - Continued

- Thus, in Kronecker Product notation

$$\begin{aligned} X_t \otimes I_{2 \times 2} &= \begin{bmatrix} 1I_{2 \times 2} & x_{1t}I_{2 \times 2} & x_{2t}I_{2 \times 2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & x_{1t} & 0 & x_{2t} & 0 \\ 0 & 1 & 0 & x_{1t} & 0 & x_{2t} \end{bmatrix} \end{aligned} \quad (6)$$

Kronecker Products - Continued

- A little out of order, I will introduce the $\text{vec}(\cdot)$ operator (Dhrymes 2000 p. 118)

$$\text{vec}(A) = \begin{bmatrix} a'_{.1} \\ a'_{.2} \\ \dots \\ a'_{.m} \end{bmatrix} \quad (7)$$

Kronecker Products - Continued

- Consider three observations from the two equation system introduced above, the dependent and independent variables could be represented as

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{bmatrix}, X = \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ 1 & x_{31} & x_{32} \end{bmatrix} \quad (8)$$

Kronecker Products - Continued

- The joint regression can be written as

$$\text{vec}(Y) = (X \otimes I_{2 \times 2}) \alpha = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \\ y_{31} \\ y_{32} \end{bmatrix} \begin{bmatrix} 1 & 0 & x_{11} & 0 & x_{12} & 0 \\ 0 & 1 & 0 & x_{11} & 0 & x_{12} \\ 1 & 0 & x_{21} & 0 & x_{22} & 0 \\ 0 & 1 & 0 & x_{21} & 0 & x_{22} \\ 1 & 0 & x_{31} & 0 & x_{32} & 0 \\ 0 & 1 & 0 & x_{31} & 0 & x_{32} \end{bmatrix} \begin{bmatrix} \alpha_{01} \\ \alpha_{02} \\ \alpha_{11} \\ \alpha_{12} \\ \alpha_{21} \\ \alpha_{22} \end{bmatrix} \quad (9)$$

Kronecker Products - Continued

- **Matrix Multiplication:** If $C_1 = A_1 \otimes B_1$ and $C_2 = A_2 \otimes B_2$ then $C_1 C_2 = A_1 A_2 \otimes B_1 B_2$ as long as the matrix multiplications are valid ($A_1 \in M_{m \times n}$ and $A_2 \in M_{n \times r}$).
- **Matrix Inversion:** If $C = A \otimes B$ (and A and B are square matrices) then $C^{-1} = A^{-1} \otimes B^{-1}$.
- **Trace [Proposition 2.34]:** $\text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B)$.
- **Determinants [Proposition 2.35]):** $|A \otimes B| = |A|^n |B|^m$

Eigenvalues and Eigenvectors

- Eigenvalues are sometimes referred to as Characteristic values or the Characteristic Roots. They are defined as

$$Ax = \lambda x \quad (10)$$

- λ is the Eigenvalue while x is referred to as the Eigenvector.
- The Eigenvalues can be derived by solving

$$|A - \lambda I| = 0 \quad (11)$$

- The Eigenvector associated with each Eigenvalue is defined by the singularity

$$(\lambda I - A)x = 0 \quad (12)$$

Eigenvalues and Eigenvectors - Continued

- To make our discussion a little more concrete, consider the X matrix for a regression

$$X = \begin{bmatrix} 0.01657 & 0.08377 \\ 0.03580 & -0.02591 \\ -0.00356 & 0.01241 \\ 0.00459 & -0.16457 \\ 0.04186 & 0.18388 \end{bmatrix}. \quad (13)$$

- Taking $X'X/5$ yields

$$\frac{X'X}{5} = \begin{bmatrix} 0.000668 & 0.001471 \\ 0.001471 & 0.013748 \end{bmatrix} \quad (14)$$

Eigenvalues and Eigenvectors - Continued

- Setting up the Eigenvalue problem

$$\begin{vmatrix} 0.000668 - \lambda & 0.001471 \\ 0.001471 & 0.013748 - \lambda \end{vmatrix} =$$

$$(0.000668 - \lambda)(0.013748 - \lambda) - (0.001471)^2 = 0 \quad (15)$$

$$\Rightarrow (0.000009 - 0.014416\lambda + \lambda^2) - 0.000002 = 0$$

- Which can be solved using the quadratic equation to yield $\lambda = (0.0005, 0.0139)$.

- Next as described in Lecture 2, we append another variable based on a random draw

$$X_{.3} = X_{.1} + X_{.2} + \sigma \epsilon \quad (16)$$

where ϵ is a random vector.

- Starting with $\sigma = 0.05$, the eigenvalues for the new $X'X$ are $\lambda = (0.0001989, 0.0014097, 0.033207)$.
- If we reduce σ by $1/2$, the eigenvalues become $\lambda = (0.00008029, 0.009089, 0.031902)$.
- Taking the limit and letting $\sigma \rightarrow 0$, the eigenvalues become $\lambda = (0, 0.006775, 0.03109)$.

- Another property of the Eigenvalue system that will be useful is apparent writing the result as

$$AX = X\Lambda \ni \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad (17)$$

and X is a matrix whose columns are Eigenvectors.

- Given that the Eigenvectors are non-singular (by definition)

$$X^{-1}AX = \Lambda \quad (18)$$

will provide the basis for a generalized inverse.

Orthogonal Matrices

- By definition, two vectors a and b are orthogonal if $a'b = 0$.
- The vectors are said to be orthonormal if $a'a = 1$ and $b'b = 1$ as well.
- **Definition 2.31:** Q (a square matrix of order m) is orthonormal if and only if its columns are orthonormal.
- **Proposition 2.46:** The transpose of orthonormal matrices are their inverse.
 - Since we define the inverse of Q (written Q^{-1}) if $Q^{-1}Q = I$, then since $Q'Q = I$ (by the definition of orthonormality) $Q^{-1} = Q'$.
- The orthonormal matrix is related to the Gram-Schmidt process which will be discussed later.

Idempotent Matrices

- An idempotent matrix is a matrix such that

$$AA = A \quad (19)$$

- One frequently used idempotent matrix in econometrics is the project matrix

$$M = I_{n \times n} - X [X'X]^{-1} X' \quad (20)$$

from the residual projection $e = y - X \left([X'X]^{-1} X'y \right)$.

- Mathematically

$$\begin{aligned}
 & \left(I - X [X'X]^{-1} X' \right) \left(I - X [X'X]^{-1} X' \right) \\
 & I - 2X [X'X]^{-1} X' + X [X'X]^{-1} X' X [X'X]^{-1} X' \\
 & I - 2X [X'X]^{-1} X' + X [X'X]^{-1} X' \\
 & I - X [X'X]^{-1} X'
 \end{aligned} \tag{21}$$

Semidefinite and Definite Matrices

- In optimization we need the matrix to be negative definite.
Matrix A is negative definite if for any vector x

$$x'Ax < 0 \quad (22)$$

- Most matrices in econometrics involve a variance matrix estimated as $\Sigma = X'X$. As such, most matrices in econometrics are either positive definite

$$x'Ax > 0 \quad (23)$$

or positive semi-definite

$$x'Ax \geq 0 \quad (24)$$

- * Khuri, A.I. 1993. *Advanced Calculus with Applications in Statistics*. John Wiley & Sons.
- * Popper, K. 2010. *The Logic of Scientific Discovery* Routledge Classics [Chapter 6: Degrees of Testability, 95-120].