

Composite Tests and the Likelihood Ratio Test: Lecture XXIV

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1 Simple Tests against a Composite

2 Composite against Composite

Simple Tests against a Composite

- Mathematically, we now can express the tests as testing between $H_0 : \theta = \theta_0$ against $H_1 : \theta \in \Theta_1$, where Θ_1 is a subset of the parameter space.
- Given this specification, we must modify our definition of the power of the test because the β value (the probability of accepting the null hypothesis when it is false) is not unique. In this regard, it is useful to develop the power function.
 - Definition 9.4.1.** If the distribution of the sample X depends on a vector of parameters θ , we define the power function of the test based on the critical region R by

$$Q(\theta) = P(X \in R | \theta) \quad (1)$$

- Definition 9.4.2.** Let $Q_1(\theta)$ and $Q_2(\theta)$ be the power functions of two tests respectively. Then we say that the first test is uniformly better (or uniformly most powerful) than the second in testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \in \Theta_1$ if $Q_1(\theta_0) = Q_2(\theta_0)$ and $Q_1(\theta) \geq Q_2(\theta)$ for all $\theta \in \Theta_1$ and $Q_1(\theta) > Q_2(\theta)$ for at least one $\theta \in \Theta_1$.

- **Definition 9.4.3.** A test R is the uniformly most power (UMP) test of size (level) α for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \in \Theta_1$ if $P(R|\theta_0) = (\leq)\alpha$ and any other test R_1 such that $P(R|\theta_0) = (\leq)\alpha$, we have $P(R|\theta) \geq P(R_1|\theta)$ for any $\theta \in \Theta_1$.
- The likelihood ratio test usually gives the UMP test if an UMP test exists. Even if the UMP does not exist, the likelihood ratio test has good properties.
 - ▶ **Definition 9.4.4.** Let $L(x|\theta)$ be the likelihood function and let the null and alternative hypotheses be $H_0 : \theta = \theta_0$ and $H_1 : \theta \in \Theta_1$, where Θ_1 is a subset of the parameter space Θ . Then the likelihood ratio test of H_0 against H_1 is defined by the critical region

$$\Lambda = \frac{L(\theta_0|x)}{\sup_{\theta_0 \cup \Theta_1} L(\theta|x)} < c \quad (2)$$

where c is chosen to satisfy $P(\Lambda < c|H_0) = \alpha$ for a certain value of α .

- Continued

- ▶ **Example 9.4.3.** Let the sample be $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2, \dots, n$ where σ^2 is assumed to be known. Let x_i be the observed value of X_i . Testing $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$. The likelihood ratio test is to reject H_0 if

$$\Lambda = \frac{\exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right]}{\sup_{\mu \geq \mu_0} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right]} < c \quad (3)$$

- Continued

- ▶ Assume that we had the sample $X = \{6, 7, 8, 9, 10\}$ from the preceding example and wanted to construct a likelihood ratio test for $\mu > 7.5$

$$\Lambda = \frac{\exp \left[-\frac{1}{2\sigma^2} \left([6 - 7.5]^2 + [7 - 7.5]^2 + \cdots [10 - 7.5]^2 \right) \right]}{\exp \left[-\frac{1}{2\sigma^2} \left([6 - 8]^2 + [7 - 8]^2 + \cdots [10 - 8]^2 \right) \right]} \quad (4)$$

where 8 is the maximum likelihood estimate of μ , assuming a standard deviation of 1.5 yields a likelihood ratio of 1.284.

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- ▶ **Theorem 9.4.1.** Let Λ be the likelihood ratio test statistic. Then, -2Λ is asymptotically distributed as chi-squared with the degrees of freedom equal to the number of exact restrictions implied by H_0 .

Composite against Composite

- **Definition 9.5.1.** A test R is the uniformly most power test of size (level) α if $\sup_{\theta \in \Theta} P(R|\theta) = (\leq)\alpha$ and for any other test R_1 such that $\sup_{\theta \in \Theta} P(R_1|\theta) = (\leq)\alpha$ we have $P(R|\theta) \geq P(R_1|\theta)$ for any $\theta \in \Theta$.
- **Definition 9.5.2.** Let $L(x|\theta)$ be the likelihood function. Then the likelihood ratio test of H_0 against H_1 is defined by the critical region

$$\Lambda = \frac{\sup_{\theta_0} L(\theta|x)}{\sup_{\theta_0 \cup \Theta_1} L(\theta|x)} < c \quad (5)$$

where c is chosen to satisfy $\sup_{\Theta} P(\Lambda < c|\theta)$ for a certain specified value of α .

- **Example 9.5.2.** Let the sample $X_i \sim N(\mu, \sigma^2)$ with unknown σ^2 , $i = 1, 2, \dots, n$. We want to test $H_0 : \mu = \mu_0$ and $0 < \sigma^2 < \infty$ against $H_1 : \mu > \mu_0$ and $0 < \sigma^2 < \infty$.

$$L[\theta] = (2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right] \quad (6)$$

Using the concentrated likelihood function at the null hypothesis

$$\sup_{\Theta_0} L(\theta) = (2\pi)^{-n/2} (\bar{\sigma}^2)^{-n/2} \exp \left[-\frac{n}{2} \right] \quad (7)$$

$$\bar{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu_0)^2$$

- The likelihood value can be compared with the maximum likelihood value of

$$\sup_{\Theta_0} L(\theta) = (2\pi)^{-n/2} (\hat{\sigma}^2)^{-n/2} \exp \left[-\frac{n}{2} \right]$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (8)$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

The critical region then becomes

$$\left(\frac{\bar{\sigma}^2}{\hat{\sigma}^2} \right)^{-\frac{n}{2}} < c \quad (9)$$

- Turning back to the Bayesian model, the Bayesian would solve the problem testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$. Let $L_2(\theta)$ the loss incurred by choosing H_0 and $L_1(\theta)$ by choosing H_1 . The Bayesian rejects H_0 if

$$\int_{-\infty}^{\infty} L_1(\theta) f(\theta|x) d\theta < \int_{-\infty}^{\infty} L_2(\theta) f(\theta|x) d\theta \quad (10)$$

where $f(\theta|x)$ is the posterior distribution of θ .