

Empirical Examples of the Central Limit Theorem: Lecture XVIII

Charles B. Moss

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Back to Asymptotic Normality

- The characteristic function of a random variable X is defined as

$$\begin{aligned}\phi_X(t) &= \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX) + i\sin(tX)] \\ &= \mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)].\end{aligned}\tag{1}$$

Note that this definition parallels the definition of the moment-generating function

$$M_X(t) = \mathbb{E}[e^{tX}]\tag{2}$$

- Like the moment-generating function there is a one-to-one correspondence between the characteristic function and the distribution of random variable. Two random variables with the same characteristic function are distributed the same.
- The characteristic function of the uniform distribution function of the uniform distribution function is

$$\phi_X(t) = e^{it} - 1 \quad (3)$$

The characteristic function of the Normal distribution function is

$$\phi_X(t) = \exp\left(it\mu - \frac{\sigma^2 t^2}{2}\right) \quad (4)$$

- The Gamma distribution function

$$f(X) = \frac{a^r x^{r-1} e^{-\alpha X}}{\Gamma(r)} \quad X \in (0, \infty) \quad (5)$$

which implies the characteristic function

$$\phi_X(t) = \frac{1}{\left(1 - \frac{it}{\alpha}\right)^r} \quad (6)$$

- Taking a Taylor series expansion of around the point $t = 0$ yields

$$\phi_z(t) = \phi_z(0) + \frac{1}{1!}\phi'_z(0)t + \frac{1}{2!}\phi''_z(0)t^2 + o(t^2)$$
$$\ni Z = \frac{X - \mu}{\sigma\sqrt{n}} \tag{7}$$

To work on this expression we note that

$$\phi_X(0) = 1 \tag{8}$$

- For any random variable X , and

$$\phi_X^{(k)}(0) = i^k E(X^k) \quad (9)$$

Putting these two results into the second-order Taylor series expansion

$$1 + E[Z]it + E[Z^2] \frac{i^2 t^2}{2} + o(t^2) = 1 - \frac{t^2}{2} + o(t^2) \quad (10)$$

$$\ni: E(Z) = 0, E(Z^2) = 1$$

- Thus,

$$\begin{aligned}\phi_z(t) &= \phi_z(0) + \frac{1}{1!} \phi'_z(0) t + \frac{1}{2!} \phi''_z(0) t^2 + o(t^2) \\ &= 1 + E[Z] it + E[Z^2] \frac{i^2 t^2}{2} + o(t^2) \\ &= 1 - \frac{t^2}{2} + o(t^2) \Leftrightarrow 1 - \frac{t^2}{2} = E(e^{iy}) \ni y \sim N(0, 1)\end{aligned}\tag{11}$$

Assume a Gamma Distribution

$$\phi_X(t) = \frac{1}{\left(1 - \frac{it}{\alpha}\right)^r} \Rightarrow \begin{cases} \phi'_X(t) = \frac{ir \left(1 - \frac{it}{\alpha}\right)^{1-r}}{\alpha} \Rightarrow \phi'_X(0) = \frac{ir}{\alpha} \\ \phi''_X(t) = \frac{(-1-r)r \left(1 - \frac{ir}{\alpha}\right)^{-2-r}}{\alpha^2} \Rightarrow \frac{(-1-r)r}{\alpha^2} \end{cases} \quad (1)$$

Wrapping Up Loose Ends

- Application of Holder's Inequality.
 - Holder's inequality

$$|E[XY]| \leq E|XY| \leq (E|X|^p)^{1/p} (E|Y|^q)^{1/q} \quad (13)$$

- **Example 4.7.1:** If X and Y have means μ_X , μ_Y and variances σ_X^2 , σ_Y^2 , respectively, we can apply the Cauchy-Schwartz Inequality (Holders inequality with $p = q = 1/2$) to get

$$E|(X - \mu_X)(Y - \mu_Y)| \leq \left\{ E[(X - \mu_X)^2]^{1/2} \right\} \left\{ E[(Y - \mu_Y)^2]^{1/2} \right\} \quad (14)$$

- Squaring both sides and substituting for variances and covariances yields

$$(\text{Cov}(X, Y))^2 \leq \sigma_X^2 \sigma_Y^2 \quad (15)$$

Which implies that the absolute value of the correlation coefficient is less than one.

Application of Chebychev's Inequality

- Chebychev's Inequality: Let X be a random variable and let $g(X)$ be a nonnegative function. Then, for any $r > 0$

$$P[g(X) \geq r] \leq \frac{E[g(X)]}{r} \quad (16)$$

- **Example 4.7.3:** The most widespread use of Chebychev's Inequality involves means and variances. Let $g(x) = (x - \mu)^2 / \sigma^2$, where $\mu = E[X]$ and $\sigma^2 = V(X)$. Let $r = t^2$

$$P\left(\frac{(X - \mu)^2}{\sigma^2} \geq t^2\right) \leq \frac{1}{t^2} E\left[\frac{(X - \mu)^2}{\sigma^2}\right] = \frac{1}{t^2} \quad (17)$$

- Since

$$\frac{(X - \mu)^2}{\sigma^2} \geq t^2 \Rightarrow (X - \mu)^2 \geq \sigma^2 t^2 \Rightarrow |X - \mu| \geq \sigma t \quad (18)$$

$$P(|X - \mu| \geq \sigma t) \leq \frac{1}{t^2}$$

- Letting $t = 2$, this becomes

$$P(|X - \mu| \geq 2\sigma) \leq \frac{1}{4} = 0.25 \quad (19)$$

However, this inequality may not say much, since for the normal distribution

$$\begin{aligned} P(|X - \mu| \geq 2\sigma) &= 2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x - \mu)^2}{2\sigma^2}\right] dx \\ &= 2 \times (0.0227) = 0.0455 \end{aligned} \quad (20)$$

Thus, the actual probability under the standard normal distribution is much smaller than the Chebychev Inequality.

- Casella and Berger offer a proof of the Central Limit Theorem (Theorem 5.3.3) based on the moment generating function instead of the characteristic function. However, they note that the proof using the characteristic function is a stronger result.

Normal Approximation of the Binomial

- Starting from the Binomial distribution function

$$b(n, r, p) = \frac{n!}{(n-r)!r!} p^r (1-p)^{n-r} \quad (21)$$

- First, assume that $n = 10$ and $p = 0.5$. The probability of $r \leq 3$ is

$$P(r \leq 3) = b(10, 0, 0.5) + b(10, 1, 0.5) + b(10, 3, 0.5) = 0.1719 \quad (22)$$

Note that this distribution has a mean of 5 and a variance of 2.5. Given this we can compute

$$z^* = \frac{3 - 5}{\sqrt{2.5}} = -1.265 \quad (23)$$

- Integrating the standard normal distribution function from negative infinity to -1.265 yields

$$P(z^* \leq -1.265) = \int_{-\infty}^{-1.265} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] dz = 0.1030. \quad (24)$$

- Expanding the sample size to 20 and examining the probability that $r \leq 6$ yields

$$P(r \leq 6) = \sum_{i=0}^6 b(20, i, 5) = 0.0577 \quad (25)$$

This time the mean of the distribution is 10 and the variance is 5. The resulting $z^* = -1.7889$. The integral of the normal distribution function from negative infinity to -1.7889 is .0368.

- As the sample size increases the binomial probability approaches the normal probability.

Reading

- Moss 7.1 – 7.3.1