

Fourier, Sobolev, and Complex Analysis: Lecture XVIIb

Charles B. Moss

October 1, 2026

- 1 A General Discussion of Expansions
- 2 Convergence
 - Sobolev Space
- 3 Complex Analysis and the Fourier Expansion

A General Discussion of Expansions

- As a starting point, consider the standard Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{n!} \frac{\partial^n f(x)}{\partial x^n} \Big|_{x \rightarrow x_0} (x - x_0)^n \right) \quad (1)$$

- The basic contention is that this expression is the result of a series. Specifically, suppose that I construct a series c_n

$$c_n = \frac{1}{n!} \left(\frac{\partial^n f(x)}{\partial x^n} \Big|_{x \rightarrow x_0} (x - x_0)^n \right) \quad (2)$$

then we can write $f(x) = \sum_{n=0}^{\infty} c_n$.

Fourier Expansion

- One concept from Herman is that we want to demonstrate that this sum converges - That there is some value L such that

$$\lim_{n \rightarrow \infty} \left| \sum_{i=1}^n c_i \left\{ = \tilde{f}(x|x_0) \right\} - L \right| < \epsilon. \quad (3)$$

Intuitively, the Taylor series expansion converges to the true underlying function value as long as c_n converges to zero.

- Turning to Herman Chapter 2, there is another expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \quad (4)$$

that yields the same type of results.

Convergence – An Overview

- The question is one of convergence and, more specifically, the type of convergence. How fast does the solution converge and how do we define convergence?
- To address this question, let me introduce something referred to as an L_p norm.

$$\|f_\alpha\| = \left(\sum_{i=1}^N |f_{\alpha i}^p| \right)^{\frac{1}{p}} \quad (5)$$

where f_α is a vector. We have seen this before.

Sobolev Space

- Next, I want to introduce the concept of a Sobolev Space. Written in its most grisly form

$$\|f_{k,p}\| = \left(\sum_{I=0}^k \|f^{(i)}\|^p \right)^{\frac{1}{p}} \quad (6)$$

which looks horrible. Let me rewrite it slightly and almost correctly

$$\|f\|_{k,n} = \left(\sum_{i=0}^n \left| \frac{\partial^i f(x)}{\partial x_i} \right|^p \right)^{\frac{1}{p}} \quad (7)$$

So the Sobolev space is basically the norm across the set of derivatives as opposed to the Taylor series which converges in the level of the function.

- There is a difference between convergences - the Taylor series convergence is “inferior” to the Sobolev convergence.

Complex Analysis and the Fourier Expansion

- As a starting point, consider the fact that restricting our analysis to real numbers is actually quite constraining – there are as many imaginary numbers as real numbers. Does this justify Complex Analysis – maybe, yes.
- By Euler's formula

$$e^{ix} = \cos(x) + i \sin(x) \quad (8)$$

bringing this together with Equation 4 we would infer

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + ib_n \sin\left(\frac{n\pi x}{L}\right) \right). \quad (9)$$

- So, we can approximate any function in complex number space using a Fourier approximation.
- The basic concept is that as we increase the number terms in Equation 9, the Fourier expansion infinitely approaches the true value of the derivatives in Sobolev space.

Approximating a Really Wicked Function

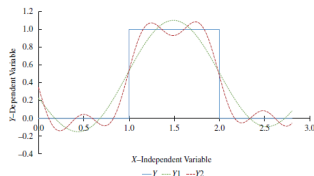


Fig. 2.10 Box-Response with First- and Second-Order Fourier Functions

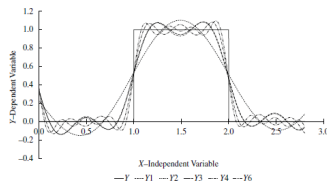


Fig. 2.11 Box-Response with Sixth-Order Fourier Function

- If we are interested in the moments (e.g., the derivatives of the moment generating and characteristic functions), the characteristic function is better: It gets the derivatives correct.