Empirical Examples of the Central Limit Theorem: Lecture XVI

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Back to Asymptotic Normality

- The characteristic function of a random variable X is defined as

$$\begin{aligned} \phi_X(t) &= \mathrm{E}\left[e^{itX}\right] = \mathrm{E}\left[\cos(tX) + i\sin(tX)\right] \\ &= \mathrm{E}\left[\cos(tX)\right] + i\mathrm{E}\left[\sin(tX)\right]. \end{aligned} \tag{1}$$

  Note that this definition parallels the definition of the moment-generating function

$$M_X(t) = \mathrm{E}\left[e^{tX}\right] \tag{2}$$

- Like the moment-generating function there is a one-to-one correspondence between the characteristic function and the distribution of random variable. Two random variables with the same characteristic function are distributed the same.
- The characteristic function of the uniform distribution function of the uniform distribution function is

$$\phi_X(t) = e^{it} - 1 \tag{3}$$

The characteristic function of the Normal distribution function is

$$\phi_X(t) = \exp\left(it\mu - \frac{\sigma^2 t^2}{2}\right) \tag{4}$$

- The Gamma distribution function

$$f(X) = \frac{a^r x^{r-1} e^{-\alpha X}}{\Gamma(r)} \quad X \in (0, \infty) \tag{5}$$

which implies the characteristic function

$$\phi_X(t) = \frac{1}{\left(1 - \frac{it}{\alpha}\right)^r} \tag{6}$$

- Taking a Taylor series expansion of around the point $t = 0$ yields

$$\begin{aligned} \phi_z(t) = \phi_z(0) + \frac{1}{1!}\phi_z'(0)\,t + \frac{1}{2!}\phi_z''(0)\,t^2 + o\left(t^2\right) \\ \ni: Z = \frac{X - \mu}{\sigma\sqrt{n}} \end{aligned} \tag{7}$$

To work on this expression we note that

$$\phi_X(0) = 1 \tag{8}$$

- For any random variable X, and

$$\phi_X^{(k)}(0) = i^k \mathrm{E}\left(X^k\right) \tag{9}$$

Putting these two results int the second-order Taylor series expansion

$$\begin{gathered} 1 + \frac{\mathrm{E}(Z)}{i} t + \frac{\mathrm{E}\left(Z^2\right)}{i^2} \frac{t^2}{2} + o\left(t^2\right) = 1 - \frac{t^2}{2} + o\left(t^2\right) \\ \ni : \mathrm{E}(Z) = 0,\ \mathrm{E}\left(Z^2\right) = 1 \end{gathered} \tag{10}$$

- Thus,

$$
\begin{aligned}
\phi_z(t) &= \phi_z(0) + \frac{1}{1!}\phi_z'(0)\,t + \frac{1}{2!}\phi_z''(0)\,t^2 + o\left(t^2\right) \\
&= 1 + \frac{\mathrm{E}(Z)}{i}t + \frac{\mathrm{E}(Z)}{i^2}\frac{t^2}{2} + o\left(t^2\right) \qquad (11) \\
&= 1 - \frac{t^2}{2} + o\left(t^2\right) \Leftrightarrow 1 - \frac{t^2}{2} = \mathrm{E}\left(e^{iy}\right) \ni: y \sim N(0,1)
\end{aligned}
$$

Wrapping Up Loose Ends

- Application of Holder's Inequality.
  - Holder's inequality

$$|\mathrm{E}[XY]| \leq \mathrm{E}|XY| \leq \left(\mathrm{E}|X|^p\right)^{1/p} \left(\mathrm{E}|Y|\right)^{1/q} \tag{12}$$

  - **Example 4.7.1**: If X and Y have means μ_X , μ_Y and variances σ_X^2 , σ_Y^2 , respectively, we can apply the Cauchy-Schwartz Inequality (Holders inequality with $p = q = 1/2$) to get

$$\mathrm{E}|(X-\mu_X)(Y-\mu_Y)| \leq \left\{ \mathrm{E}\left[(X-\mu_x)^2\right]^{1/2} \right\} \left\{ \mathrm{E}\left[(Y-\mu_y)^2\right] \right\}^{1/2} \tag{13}$$

Squaring both sides and substituting for variances and covariances yields

$$\left(\mathrm{Cov}(X,Y)\right)^2 \leq \sigma_X^2 \sigma_Y^2 \tag{14}$$

Which implies that the absolute value of the correlation coefficient is less than one.

Application of Chebychev's Inequality

- Chebychev's Inequality: Let X be a random variable and let $g(X)$ be a nonnegative function. Then, for any $r > 0$

$$P[g(X) \geq r] \leq \frac{rmE[g(X)]}{r} \tag{15}$$

- **Example 4.7.3**: The most widespread use of Chebychev's Inequality involves means and variances. Let $g(x) = (x-\mu)^2/\sigma^2$, where $\mu = \mathrm{E}[X]$ and $\sigma^2 = V(X)$. Let $r = t^2$

$$P\left(\frac{(X-\mu)^2}{\sigma^2} \geq t^2\right) \leq \frac{1}{t^2}\mathrm{E}\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{t^2} \tag{16}$$

- Since

$$\frac{(X-\mu)^2}{\sigma^2} \geq t^2 \Rightarrow (X-\mu)^2 \geq \sigma^2 t^2 \Rightarrow |X-\mu| \geq \sigma t$$
$$P\left(|X-\mu| \geq \sigma t\right) \leq \frac{1}{t^2} \tag{17}$$

- Letting $t = 2$, this becomes

$$P\left(|X - \mu| \geq 2\sigma\right) \leq \frac{1}{4} = 0.25 \tag{18}$$

However, this inequality may not say much, since for the normal distribution

$$\begin{aligned} P\left(|X - \mu| \geq 2\sigma\right) &= 2\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \\ &= 2 \times (0.0227) = 0.0455 \end{aligned} \tag{19}$$

Thus, the actual probability under the standard normal distribution is much smaller than the Chebychev Inequality.

- Casella and Berger offer a proof of the Central Limit Theorem (Theorem 5.3.3) based on the moment generating function instead of the characteristic function. However, they note that the proof using the characteristic function is a stronger result.

Normal Approximation of the Binomial

- Starting from the Binomial distribution function

$$b(n,r,p) = \frac{n!}{(n-r)!r!} p^r (1-p)^{n-r} \tag{20}$$

- First, assume that $n = 10$ and $p = 0.5$. The probability of $r \leq 3$ is

$$P(r \leq 3) = b(10,0,0.5) + b(10,1,0.5) + b(10,3,0.5) = 0.1719 \tag{21}$$

Note that this distribution has a mean of 5 and a variance of 2.5. Given this we can compute

$$z^* = \frac{3-5}{\sqrt{2.5}} = -1.265 \tag{22}$$

- Integrating the standard normal distribution function from negative infinity to -1.265 yields

$$P\left(z^{*} \leq -1.265\right) = \int_{-\infty}^{-1.265} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^{2}}{2}\right] dz = 0.1030. \qquad (23)$$

- Expanding the sample size to 20 and examining the probability that $r \leq 6$ yields

$$P\left(r \leq 6\right) = \sum_{i=0}^{6} b\left(20, i, 5\right) = 0.0577 \qquad (24)$$

  This time the mean of the distribution is 10 and the variance is 5. The resulting $z^{*} = -1.7889$. The integral of the normal distribution function from negative infinity to -1.7889 is .0368.

- As the sample size declines the binomial probability approaches the normal probability.