

Binomial and Normal Random Variables: Lecture XIII

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September 16, 2026

Reading Problem

- Derive the moment generating function for

$$f(x) = \frac{2e^{2-2x}}{e^2 - 1}, \quad x \in [0, 1]. \quad (1)$$

1 Bernoulli Random Variables

2 Binomial Random Variables

- Pascal's Triangle

Bernoulli Random Variables

- The Bernoulli distribution characterizes the coin toss. Specifically, there are two events $X = 0, 1$ with $X = 1$ occurring with probability p . The probability distribution function $P[X]$ can be written as

$$P[X] = p^x (1 - p)^{1-x} \quad (2)$$

- Next, we need to develop the probability of $X + Y$ where both X and Y are identically distributed. If the two events are independent, the probability becomes

$$\begin{aligned} P[X, Y] &= P[X] P[Y] = p^x p^y (1 - p)^{1-x} (1 - p)^{1-y} \\ &= p^{x+y} (1 - p)^{2-x-y} \end{aligned} \quad (3)$$

- Now, this density function is only concerned with three outcomes $Z = X + Y = \{0, 1, 2\}$.
 - There is only one way each for $Z = 0$ or $Z = 2$.
 - Specifically for $Z = 0$, $X = 0$ and $Y = 0$.
 - Similarly, for $Z = 2$, $X = 1$ and $Y = 1$.
 - However, for $Z = 1$ either $X = 1$ and $Y = 0$ or $X = 0$ and $Y = 1$. Thus, we can derive

$$\begin{aligned}P[Z = 0] &= p^0 (1 - p)^{2-0} \\P[Z = 1] &= P[X = 1, Y = 0] + P[X = 0, Y = 1] \\&= p^{1+0} (1 - p)^{2-1-0} + p^{0+1} (1 - p)^{2-0-1} \\&= 2p^1 (1 - p)^1 \\P[Z = 2] &= p^2 (1 - p)^0\end{aligned}\tag{4}$$

- Next we expand the distribution to three independent Bernoulli events where $Z = W + X + Y = \{0, 1, 2, 3\}$.

$$\begin{aligned} P[Z] &= P[W, X, Y] \\ &= P[W] P[X] P[Y] \\ &= p^w p^x p^y (1-p)^{1-w} (1-p)^{1-x} (1-p)^{1-y} \\ &= p^{w+x+y} (1-p)^{3-w-x-y} \\ &= p^z (1-p)^{3-z} \end{aligned} \tag{5}$$

- Again, there is only one way for $Z = 0$ and $Z = 3$.
- However, there are now three ways for $Z = 1$ or $Z = 2$.
Specifically, $Z = 1$ if $W = 1, X = 1$ or $Y = 1$. In addition, $Z = 2$ if $W = 1, X = 1$, and $Y = 0$, or if $W = 1, X = 0$, and $Y = 1$, or if $W = 0, X = 1$, and $Y = 1$.
- Thus the general distribution function for Z can now be written as

$$\begin{aligned}
 P[Z = 0] &= p^0 (1 - p)^{3-0} \\
 P[Z = 1] &= p^{1+0+0} (1 - p)^{3-1-0-0} + p^{0+1+0} (1 - p)^{3-0-1-0} + \\
 &\quad p^{0+0+1} (1 - p)^{3-0-0-1} = 3p^1 (1 - p)^2 \\
 P[Z = 2] &= p^{1+1+0} (1 - p)^{3-1-1-0} + p^{1+0+1} (1 - p)^{3-1-0-1} + \\
 &\quad p^{0+1+1} (1 - p)^{3-0-1-1} = 3p^2 (1 - p)^1 \\
 P[Z = 3] &= p^3 (1 - p)^0
 \end{aligned}$$

(6)

Binomial Random Variables

- Based on this development, the binomial distribution can be generalized as the sum of n Bernoulli events

$$P[Z = r] = \binom{n}{r} p^r (1 - p)^{n-r} \quad (7)$$
$$C_r^n = \binom{n}{r} = \frac{n!}{r! (n - r)!}.$$

where C_r^n is the combinatorial of n and r . Graphically the combinatorial can be depicted as Pascal's triangle in Figure 1. To develop the relationship between the combinatorial and the binomial function, consider the polynomial form

Pascal's Triangle

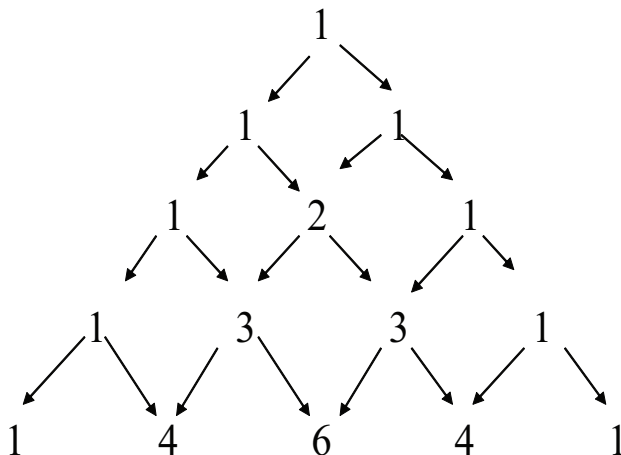


Figure: Pascal's Triangle

- Continued

$$\begin{aligned}(a+b)^1 &= a+b \\(a+b)^2 &= a^2 + 2ab + b^2 \\(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}\tag{8}$$

The general form of the polynomial can then be written as

$$(a+b)^n = \sum_{r=1}^n C_r^n a^r b^{n-r}\tag{9}$$

- As a first step, consider changing the polynomial from $a + b$ to $a - b$. Hence, $(a - b)^n$ can be written as

$$\begin{aligned}(a - b)^n &= (a + (-1)b)^n \\&= \sum_{r=1}^n C_r^n a^r (-1)^{n-r} b^{n-r} \\&= \sum_{r=1}^n (-1)^{n-r} C_r^n a^r b^{n-r}\end{aligned}\tag{10}$$

- This sequence can be linked to our discussion of the Bernoulli system by letting $a = p$ and $b = 1 - p$. Thus, the binomial distribution $X \sim B(n, p)$ can be written as

$$P(X = k) = C_k^n p^k (1 - p)^{n-k} \quad (11)$$

- For the case above, $n = 3$. The distribution function (ignoring the constants) can be written as

$$\begin{aligned} P[Z = 0] &= \binom{3}{0} p^0 (1 - p)^{3-0} \\ P[Z = 1] &= \binom{3}{1} p^1 (1 - p)^{3-1} \\ P[Z = 2] &= \binom{3}{2} p^2 (1 - p)^{3-2} \\ P[Z = 3] &= \binom{3}{3} p^3 (1 - p)^{3-3} \end{aligned} \quad (12)$$

- Next recalling Theorem 4.1.6: $E[aX + bY] = aE[X] + bE[Y]$, the expectation of the binomial distribution function can be recovered from the Bernoulli distributions

$$\begin{aligned} E[X] &= \sum_{k=0}^n C_k^n p^k (1-p)^{n-k} k \\ &= \sum_{i=1}^n \left[\sum_{X_i} p^{X_i} (1-p)^{1-X_i} X_i \right] \\ &= \sum_{i=1}^n \left[p^1 (1-p)^0 (1) + p^0 (1-p)^1 (0) \right] \\ &= \sum_{i=1}^n p = np \end{aligned} \tag{13}$$

- In addition, by Theorem 4.3.3

$$V\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n V(X_i). \quad (14)$$

Thus, variance of the binomial is simply the sum of the variances of the Bernoulli distributions or n times the variance of a single Bernoulli distribution

$$\begin{aligned} V(X) &= E[X^2] - (E[X])^2 \\ &= \left[p^1 (1-p)^0 (1^2) + p^0 (1-p)^1 (0^2) \right] - p^2 \\ &= p - p^2 = p(1-p) \\ V\left(\sum_{i=1}^n X_i\right) &= np(1-p) \end{aligned} \quad (15)$$

Reading Assignments

- Moss 5.2 and 5.3
- Popper Chapter 8 – Probability (it is a long chapter)