

Moment Generating Functions: Lecture XII

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Reading

- Derive the first, second and third moments of the distribution

$$f(x) = 2 - 2x \quad (1)$$

where $0 \leq x \leq 1$.

1 Moment Generating Function

Moment Generating Function

- Definition 2.3.3.** Let X be a random variable with cumulative distribution function $F(X)$. The moment generating function (mgf) of X (or $F(X)$), denoted $M_X(t)$, is

$$M_X(t) = E[e^{tX}] \quad (2)$$

provided that the expectation exists for t in some neighborhood of 0. That is, there is an $h > 0$ such that, for all t in $-h < t < h$, $E[e^{tX}]$ exists.

- If the expectation does not exist in a neighborhood of 0, we say that the moment generating function does not exist.
- The moment generating function can be defined as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \text{ for continuous random variables, and}$$

$$M_x(t) = \sum_x e^{tx} P[X = x] \text{ for discrete random variables}$$

- **Theorem 2.3.2** If X has *mgf* $M_X(t)$, then

$$E[X^n] = M_X^{(n)}(0) \quad (4)$$

where we define

$$M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t) \Big|_{t \rightarrow 0} \quad (5)$$

- First note that e^{tX} can be approximated around zero using a Taylor series expansion

$$M_X(t) = E[e^{tx}] = E\left[e^0 + te^{t0}(x-0) + \frac{1}{2}t^2e^{t0}(x-0)^2 + \frac{1}{6}t^3e^{t0}(x-0)^3 + \dots\right] \quad (6)$$

$$= 1 + E[x]t + E[x^2]\frac{t^2}{2} + E[x^3]\frac{t^3}{6} + \dots$$

- Note for any moment n

$$M_x^{(n)}(t) = \frac{d^n}{dt^n} M_X(t) = E[x^n] + tE[x^{n+1}] + \frac{t^2}{2}E[x^{n+2}] + \dots \quad (7)$$

- Thus, as $t \rightarrow 0$

$$M_x^{(n)}(0) = E[x^n] \quad (8)$$

- Leibnitz's Rule: If $f(x, \theta)$, $a(\theta)$, and $b(\theta)$ are differentiable with respect to θ , then

$$\begin{aligned} \frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} f(x, \theta) dx &= -f(a(\theta), \theta) \frac{d}{d\theta} a(\theta) + f(b(\theta), \theta) \frac{d}{d\theta} b(\theta) \\ &\quad + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial \theta} f(x, \theta) dx \end{aligned} \tag{9}$$

- Berger and Casella proof: Assume that we can differentiate under the integral using Leibnitz's rule, we have

$$\begin{aligned} \frac{d}{dt} M_X(t) &= \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ &= \int_{-\infty}^{\infty} \left(\frac{d}{dt} e^{tx} \right) f(x) dx \\ &= \int_{-\infty}^{\infty} x e^{tx} f(x) dx \end{aligned} \tag{10}$$

- Letting $t \rightarrow 0$, this integral simply becomes

$$\int_{-\infty}^{\infty} x f(x) dx = E[x] \quad (11)$$

- This proof can be extended for any moment of the distribution function.

Moment Generating Functions – The Uniform Distribution

- Application to the Uniform Distribution

$$M_X(t) = \int_a^b \frac{e^{tx}}{b-a} dx = \frac{1}{b-a} \frac{1}{t} \left(e^{tx} \Big|_a^b \right) = \frac{e^{bt} - e^{at}}{t(b-a)} \quad (12)$$

- Following the expansion developed earlier, we have

$$\begin{aligned} M_X(t) &= \frac{(1-1) + (b-a)t + \frac{1}{2}(b^2-a^2)t^2 + \frac{1}{6}(b^3-a^3)t^3 + \dots}{(b-a)t} \\ &= 1 + \frac{(b^2-a^2)t^2}{2(b-a)t} + \frac{(b^3-a^3)t^3}{6(b-a)t} + \dots \\ &= 1 + \frac{1}{2} \frac{(b-a)(b+a)}{(b-a)} \frac{t^2}{t} + \frac{1}{6} \frac{(b-a)(b^2+ab+a^2)}{(b-a)} \frac{t^3}{t} + \dots \\ &= 1 + \frac{1}{2}(a+b)t + \frac{1}{6}(a^2+ab+b^2)t^2 + \dots \end{aligned} \quad (13)$$

- Letting $b = 1$ and $a = 0$, the last expression becomes

$$M_X(t) = 1 + \frac{1}{2}t + \frac{1}{6}t^2 + \frac{1}{24}t^3 + \dots \quad (14)$$

- The first three moments of the uniform distribution are then

$$M_X^{(1)} = \frac{1}{2}$$

$$M_X^{(2)} = \frac{1}{6}2 = \frac{1}{3} \quad (15)$$

$$M_X^{(3)} = \frac{1}{24}6 = \frac{1}{4}$$

Moment Generating Functions – The Normal Distribution

- Application to the Univariate Normal Distribution

$$\begin{aligned}
 M_X(t) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} dx \\
 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[tx - \frac{1}{2} \frac{(x-\mu)^2}{\sigma^2} \right] dx
 \end{aligned} \tag{16}$$

- Focusing on the term in the exponent, we have

$$\begin{aligned}
 tx - \frac{1}{2} \frac{(x-\mu)^2}{\sigma^2} &= -\frac{1}{2} \frac{(x-\mu)^2 - 2tx\sigma^2}{\sigma^2} \\
 &= -\frac{1}{2} \frac{x^2 - 2x\mu + \mu^2 - 2tx\sigma^2}{\sigma^2} \\
 &= -\frac{1}{2} \frac{x^2 - 2(x\mu + tx\sigma^2) + \mu^2}{\sigma^2} \\
 &= -\frac{1}{2} \frac{x^2 - 2x(\mu + t\sigma^2) + \mu^2}{\sigma^2}
 \end{aligned} \tag{17}$$

- The next state is to complete the square in the numerator

$$\begin{aligned}x^2 - 2x(\mu + t\sigma^2) + \mu^2 + c &= 0 \\(x - (\mu + t\sigma^2))^2 &= 0 \\x^2 - 2x(\mu + t\sigma) + \mu^2 + 2t\sigma^2\mu + t^2\sigma^4 &= 0 \\c &= 2t\sigma^2\mu + t^2\sigma^4\end{aligned}\tag{18}$$

- The complete expression then becomes

$$\begin{aligned} tx - \frac{1}{2} \frac{(x - \mu)^2}{\sigma^2} &= -\frac{1}{2} \frac{(x - \mu - t\sigma^2)^2 - 2\mu\sigma^2 t - \sigma^4 t^2}{\sigma^2} \\ &= -\frac{1}{2} \frac{(x - \mu - t\sigma^2)^2}{\sigma^2} + \mu t + \frac{1}{2} \sigma^2 t^2 \end{aligned} \quad (19)$$

The moment generating function then becomes

$$\begin{aligned} M_X(t) &= \exp\left(\mu t + \frac{1}{2} \sigma^2 t^2\right) \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \frac{(x - \mu - t\sigma^2)^2}{\sigma^2}\right) dx \\ &= \exp\left(\mu t + \frac{1}{2} \sigma^2 t^2\right) \end{aligned} \quad (20)$$

Taking the first derivative with respect to t , we get

$$M_X^{(1)}(t) = (\mu + \sigma^2 t) \exp\left(\mu t + \frac{1}{2} \sigma^2 t^2\right). \quad (21)$$

- Letting $t \rightarrow 0$, this becomes

$$M_X^{(0)} = \mu \quad (22)$$

The second derivative of the moment generating function with respect to t yields

$$M_X^{(2)}(t) = \sigma^2 \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) + (\mu + \sigma^2 t)(\mu + \sigma^2 t) \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \quad (23)$$

Again, letting $t \rightarrow 0$ yields

$$M_X^{(2)}(0) = \sigma^2 + \mu^2 \quad (24)$$

- Let X and Y be independent random variables with moment generating functions $M_X(t)$ and $M_Y(t)$. Consider their sum $Z = X + Y$ and its moment generating function

$$\begin{aligned} M_Z(t) &= \mathbb{E}[e^{tz}] = \mathbb{E}[e^{t(x+y)}] = \mathbb{E}[e^{tx} e^{ty}] = \\ &\mathbb{E}[e^{tx}] \mathbb{E}[e^{ty}] = M_X(t) M_Y(t) \end{aligned} \quad (25)$$

- We conclude that the moment generating function for two independent random variables is equal to the product of the moment generating functions of each variable.
- Skipping ahead slightly, the multivariate normal distribution function can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}} |\Sigma| \exp\left(-\frac{1}{2} (x - \mu)' \Sigma^{-1} (x - \mu)\right) \quad (26)$$

- In order to derive the moment generating function, we now need a vector $\tilde{t}$. The moment generating function can then be defined as

$$M_{\tilde{X}}(\tilde{t}) = \exp\left(\mu'\tilde{t} + \frac{1}{2}\tilde{t}'\Sigma\tilde{t}\right) \quad (27)$$

- Normal variables are independent if the variance matrix is a diagonal matrix.
- Note that if the variance matrix is diagonal, the moment generating function for the normal can be written as

$$\begin{aligned} M_{\tilde{X}}(\tilde{t}) &= \exp\left(\mu'\tilde{t} + \frac{1}{2}\tilde{t}'\begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{pmatrix}\tilde{t}\right) \\ &= \exp\left(\mu_1 t_1 + \mu_2 t_2 + \mu_3 t_3 + \frac{1}{2}(t_1^2 \sigma_1^2 + t_2^2 \sigma_2^2 + t_3^2 \sigma_3^2)\right) \\ &= \exp\left(\left(\mu_1 t_1 + \frac{1}{2} t_1^2 \sigma_1^2\right) + \left(\mu_2 t_2 + \frac{1}{2} t_2^2 \sigma_2^2\right) + \left(\mu_3 t_3 + \frac{1}{2} t_3^2 \sigma_3^2\right)\right) \\ &= M_{X_1}(t) M_{X_2}(t) M_{X_3}(t) \end{aligned} \quad (28)$$

Readings

- Moss 5.1
- Popper 6 Degrees of Testability