Normal Random Variables: Lecture XII

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1. Univariate Normal Distribution
   - Mean
   - Variance

2. Normal and Binomial Distribution

Univariate Normal Distribution

- **Definition 5.2.1.** The normal density is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right] \quad -\infty < x < \infty,\ \ \sigma > 0 \tag{1}$$

- **Theorem 5.2.1** Let X be $N\left(\mu,\sigma^2\right)$ as defined in Definition 5.2.1, then $\mathrm{E}[X] = \mu$ and $V[X] = \sigma^2$.
  - Starting with the definition of the expectation

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} x \exp\left[-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right] dx \tag{2}$$

- Using the change in variables technique, we create a new random variable z such that

$$\begin{gathered} z = \frac{x - \mu}{\sigma} \Rightarrow x = z\sigma + \mu \\ dx = \sigma dz \end{gathered} \tag{3}$$

- Substituting into the original integral yields

$$\begin{aligned}\mathrm{E}[X] &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} (z\sigma + \mu) \exp\left[-\frac{1}{2}z^2\right] dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}z^2\right] + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}z^2\right] dz\end{aligned} \tag{4}$$

Taking the integral of the first term first, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} z\sigma^2 \exp\left[-\frac{1}{2}z^2\right] dx &= C \int_{-\infty}^{\infty} z \exp\left[-\frac{1}{2}z^2\right] dz \\ &= C \left(- \exp\left[-\frac{1}{2}z^2\right] \Big|_{-\infty}^{\infty} \right. = 0.\end{aligned} \tag{5}$$

The value of the second integral becomes μ by polar integration (see Lecture 5 notes).

Variance

- The variance of the normal is similarly defined except that the initial integral now becomes

$$
\begin{aligned}
V[X] &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2\exp\left[-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right]dx \\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(z\sigma+\mu-\mu)^2\exp\left[-\frac{1}{2}z^2\right]\sigma\,dz \qquad (6)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^2\sigma^2\exp\left[-\frac{1}{2}z^2\right]dz
\end{aligned}
$$

- This formulation is then completed using integration by parts

$$\begin{aligned} u = -z &\Rightarrow d = -1 \\ dv = -z\exp\left[-\frac{1}{2}z^2\right] &\Rightarrow v = \exp\left[-\frac{1}{2}z^2\right] \\ \int_{-\infty}^{\infty} zz\exp\left[-\frac{1}{2}z^2\right]dz = -\left(z\exp\left[-\frac{1}{2}z^2\right]\right|_{-\infty}^{\infty} &+ \int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}z^2\right]dz \end{aligned} \tag{7}$$

The first term of the integration by parts is clearly zero while the second is defined by polar integral. Thus,

$$V[X] = 0 + \sigma^2\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}z^2\right] = \sigma^2 \tag{8}$$

- **Theorem 5.2.2.** Let X be distributed $N\left(\mu,\sigma^2\right)$ and let $Y=\alpha+\beta Y$. Then $Y\sim N\left(\alpha+\beta\mu,\beta^2\sigma^2\right)$.
- This theorem can be demonstrated using Theorem 3.6.1 (the theorem on changes in variables)

$$g\left(y\right)=f\left[\phi^{-1}\left(y\right)\right]\left|\frac{d\phi^{-1}\left(y\right)}{dy}\right|. \tag{9}$$

In this case

$$\begin{aligned}\phi\left(x\right)=\alpha+\beta x\Leftrightarrow\phi^{-1}\left(y\right)=\frac{y-\alpha}{\beta}\\ \frac{d\phi^{-1}\left(y\right)}{dy}=\frac{1}{\beta}\end{aligned} \tag{10}$$

- The transformed normal then becomes

$$g(y) = \frac{1}{\sigma|\beta|\sqrt{2\pi}} \exp\left[-\frac{1}{2}\frac{\left(\frac{y-\alpha}{\beta}-\mu\right)^2}{\sigma^2}\right] \tag{11}$$

$$= \frac{1}{\sigma|\beta|\sqrt{2\pi}} \exp\left[-\frac{1}{2}\frac{(y-\alpha-\beta\mu)^2}{\sigma^2\beta^2}\right]$$

- Note that probabilities can be derived for any normal based on the standard normal integral. Specifically, in order to find the probability that $X \sim N(10, 4)$ lies between 4 and 8 ($P[4 < X < 8]$) implies

$$P[4 < X < 8] = P[X < 8] - P[X < 4] \tag{12}$$

Transforming each boundary to standard normal space

$$\begin{aligned} z_1 &= \frac{x_1 - 10}{2} = \frac{-6}{2} = -3 \\ z_2 &= \frac{x_2 - 10}{2} = \frac{-2}{2} = -1 \end{aligned} \tag{13}$$

- Thus, the equivalent boundary becomes

$$P[-3 < z < -1] = P[z < -1] - P[z < -3] \tag{14}$$

where z is a standard normal variable. These values can be found in a standard normal table as $P[z < -1] = 0.1587$ and $P[z < -3] = 0.0013$.

Linking the Normal Distribution to the Binomial

- In subsequent chapters we will develop the notion of the central limit theorem in more detail. However, in introducing the normal and binomial distribution functions its important to link the two together.
- As a first step, I want to graph two distributions with the same mean and variance. Specifically, I want to start with a binomial distribution with probability .5 and 20 draws. This distribution has a mean of 10 and a variance of 5. Graphically, this distribution is depicted in Figure 1. The binomial extending to 100 draws is presented in Figure 2.

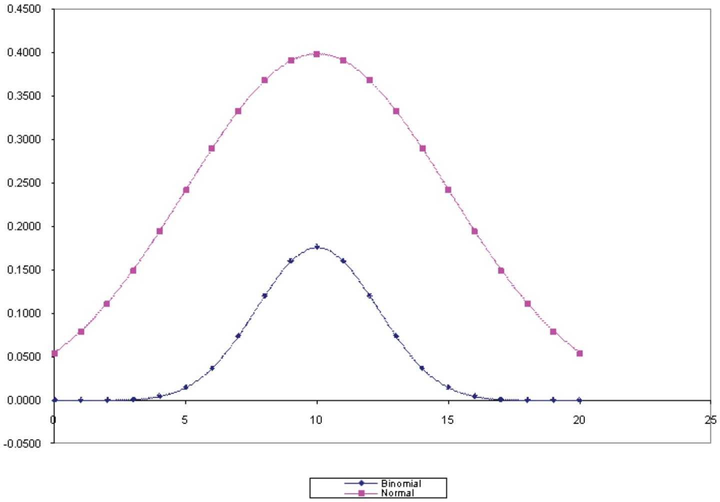

Figure: Binomial with 50 Draws and Normal

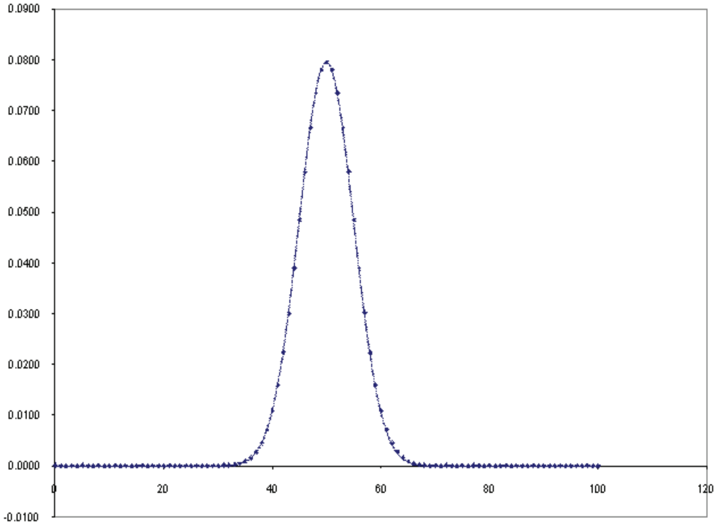

Figure: Binomial with 100 Draws