Moment Generating Functions: Lecture X

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1. Moment Generating Function

Moment Generating Function

- **Definition 2.3.3.** Let X be a random variable with cumulative distribution function $F(X)$. The moment generating function (mgf) of X (or $F(X)$), denoted $M_X(t)$, is

$$M_X(t) = \mathrm{E}\left[e^{tX}\right] \tag{1}$$

provided that the expectation exists for t in some neighborhood of 0. That is, there is an $h > 0$ such that, for all t in $-h < t < h$, $\mathrm{E}\left[e^{tX}\right]$ exists.

  - If the expectation does not exist in a neighborhood of 0, we say that the moment generating function does not exist.
  - More explicitly, the moment generating function can be defined as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \text{ for continuous random variables, and}$$

- **Theorem 2.3.2** If X has *mgf* $M_X(t)$, then

$$\mathrm{E}\left[X^n\right] = M_x^{(n)}(0) \tag{3}$$

where we define

$$M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\Big|_{t\to 0} \tag{4}$$

- First note that e^{tX} can be approximated around zero using a Taylor series expansion

$$\begin{gathered} M_X(t) = \mathrm{E}\left[e^{tx}\right] = \mathrm{E}\Big[e^0 + te^{t0}(x-0) + \\ \frac{1}{2}t^2 e^{t0}(x-0)^2 + \frac{1}{6}t^3 e^{t0}(x-0)^3 + \cdots\Big] \end{gathered} \tag{5}$$

$$= 1 + \mathrm{E}[x]\,t + \mathrm{E}\left[x^2\right]\frac{t^2}{2} + \mathrm{E}\left[x^3\right]\frac{t^3}{6} + \cdots$$

- Note for any moment n

$$M_x^{(n)}(t) = \frac{d^n}{dt^n} M_X(t) = \mathrm{E}[x^n] + \mathrm{E}\left[x^{n+1}\right] + \mathrm{E}\left[x^{n+2}\right] + \cdots \tag{6}$$

- Thus, as $t \to 0$

$$M_x^{(n)}(0) = \mathrm{E}[x^n] \tag{7}$$

- Continued
  - Leibnitz's Rule: If $f(x,\theta)$, $a(\theta)$, and $b(\theta)$ are differentiable with respect to θ , then

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx = -f(b(\theta),\theta)\frac{d}{d\theta}a(\theta) + f(a(\theta))\frac{d}{d\theta}b(\theta) \\ \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta} f(x,\theta)\,dx \tag{8}$$

  - Berger and Casella proof: Assume that we can differentiate under the integral using Leibnitz's rule, we have

$$\begin{aligned} \frac{d}{dt}M_X(t) &= \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx} f(x)\,dx \\ &= \textstyle\int_{-\infty}^{\infty}\left(\frac{d}{dt}e^{tx}\right) f(x)\,dx \\ &= \textstyle\int_{-\infty}^{\infty} x e^{tx} f(x)\,dx \end{aligned} \tag{9}$$

- Letting $t \to 0$, this integral simply bgecomes

$$\int_{-\infty}^{\infty} xf(x)\,dx = \mathrm{E}[x] \tag{10}$$

- This proof can be extended for any moment of the distribution function.

Moment Generating Functions – The Uniform Distribution

- Application to the Uniform Distribution

$$M_X(t) = \int_a^b \frac{e^{tx}}{b-a} dx = \frac{1}{b-a}\frac{1}{t}\left(e^{tx}\right|_a^b = \frac{e^{bt}-e^{at}}{t(b-a)} \qquad (11)$$

- Following the expansion developed earlier, we have

$$\begin{aligned} M_X(t) &= \frac{(1-1)+(b-a)t+\frac{1}{2}\left(b^2-a^2\right)t^2+\frac{1}{6}\left(b^3-a^3\right)t^3+\cdots}{(b-a)t} \\ &= 1+\frac{\left(b^2-a^2\right)t^2}{2(b-a)t}+\frac{\left(b^3-a^3\right)t^3}{6(b-a)t}+\cdots \qquad (12) \\ &= 1+\frac{1}{2}\frac{(b-a)(b+a)}{(b-a)}\frac{t^2}{t}+\frac{1}{6}\frac{(b-a)\left(b^2+ab+a^2\right)}{(b-a)}\frac{t^3}{t}+\cdots \\ &= 1+\tfrac{1}{2}(a+b)t+\tfrac{1}{6}\left(a^2+ab+b^2\right)t^2+\cdots \end{aligned}$$

- Letting $b = 1$ and $a = 0$, the last expression becomes

$$M_X(t) = 1 + \frac{1}{2}t + \frac{1}{6}t^2 + \frac{1}{24}t^3 + \cdots \tag{13}$$

- The first three moments of the uniform distribution are then

$$\begin{aligned} M_X^{(1)} &= \tfrac{1}{2} \\ M_X^{(2)} &= \tfrac{1}{6}2 = \tfrac{1}{3} \\ M_X^{(3)} &= \tfrac{1}{24}6 = \tfrac{1}{4} \end{aligned} \tag{14}$$

Moment Generating Functions – The Normal Distribution

- Application to the Univariate Normal Distribution

$$\begin{aligned} M_X(t) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[tx - \frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right] dx \end{aligned} \tag{15}$$

- Focusing on the term in the exponent, we have

$$\begin{aligned} tx - \frac{1}{2}\frac{(x-\mu)^2}{\sigma^2} &= -\frac{1}{2}\frac{(x-\mu)^2 - 2tx\sigma^2}{\sigma^2} \\ &= -\frac{1}{2}\frac{x^2 - 2x\mu + \mu^2 - 2tx\sigma^2}{\sigma^2} \\ &= -\frac{1}{2}\frac{x^2 - 2\left(x\mu + tx\sigma^2\right) + \mu^2}{\sigma^2} \\ &= -\frac{1}{2}\frac{x^2 - 2x\left(\mu + t\sigma^2\right) + \mu^2}{\sigma^2} \end{aligned} \tag{16}$$

- The next state is to complete the square in the numerator

$$\begin{gathered}
x^2 - 2x\left(\mu + t\sigma^2\right) + \mu^2 + c = 0 \\
\left(x - \left(\mu + t\sigma^2\right)\right)^2 = 0 \\
x^2 - 2x\left(\mu + t\sigma\right) + \mu^2 + 2t\sigma^2\mu + t^2\sigma^4 = 0 \\
c = 2t\sigma^2\mu + t^2\sigma^4
\end{gathered} \tag{17}$$

- The complete expression then becomes

$$tx - \frac{1}{2}\frac{(x-\mu)^2}{\sigma^2} = -\frac{1}{2}\frac{\left(x-\mu-t\sigma^2\right) - 2\mu\sigma^2 t - \sigma^4 t^2}{\sigma^2} \\ = -\tfrac{1}{2}\frac{\left(x-\mu-t\sigma^2\right)}{\sigma^2} + \mu t + \tfrac{1}{2}\sigma^2 t^2 \tag{18}$$

The moment generating function then becomes

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{\left(x-\mu-t\sigma^2\right)}{\sigma^2}\right)dx \\ = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \tag{19}$$

Taking the first derivative with respect to t, we get

$$M_X^{(1)}(t) = \left(\mu + \sigma^2 t\right)\exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right). \tag{20}$$

- Letting $t \rightarrow 0$, this becomes

$$M_X^{(0)} = \mu \tag{21}$$

The second derivative of the moment generating function with respect to t yields

$$\begin{gathered} M_X^{(2)}(t) = \sigma^2 \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) + \\ \left(\mu + \sigma^2 t\right)\left(\mu + \sigma^2 t\right) \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \end{gathered} \tag{22}$$

Again, letting $t \rightarrow 0$ yields

$$M_X^{(2)}(0) = \sigma^2 + \mu^2 \tag{23}$$

- Let X and Y be independent random variables with moment generating functions $M_X(t)$ and $M_Y(t)$. Consider their sum $Z = X + Y$ and its moment generating function

$$M_Z(t) = \mathrm{E}\left[e^{tz}\right] = \mathrm{E}\left[e^{t(x+y)}\right] = \mathrm{E}\left[e^{tx}e^{ty}\right] = \\ \mathrm{E}\left[e^{tx}\right]\mathrm{E}\left[e^{ty}\right] = M_X(t)\,M_Y(t) \tag{24}$$

  - We conclude that the moment generating function for two independent random variables is equal to the product of the moment generating functions of each variable.
  - Skipping ahead slightly, the multivariate normal distribution function can be written as

$$f(x) = \frac{1}{\sqrt{2\pi}}|\Sigma|\exp\left(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right) \tag{25}$$

- In order to derive the moment generating function, we now need a vector $\tilde{t}$. The moment generating function can then be defined as

$$M_{\tilde{X}}\left(\tilde{t}\right) = \exp\left(\mu'\tilde{t} + \frac{1}{2}\tilde{t}'\Sigma\tilde{t}\right) \tag{26}$$

  - Normal variables are independent if the variance matrix is a diagonal matrix.
  - Note that if the variance matrix is diagonal, the moment generating function for the normal can be written as

$$\begin{aligned} M_{\tilde{X}}\left(\tilde{x}\right) = \exp\left(\mu'\tilde{t} + \frac{1}{2}\tilde{t}'\begin{pmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^3 \end{pmatrix}\tilde{t}\right) \\ = \exp\left(\mu_1 t_1 + \mu_2 t_1 + \mu_3 t_3 + \frac{1}{2}\left(t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + t_3^2\sigma_3^2\right)\right) \\ = \exp\left(\left(\mu_1 t_1 + \frac{1}{2}t_1^2\sigma_1^2 t_1^2\right) + \left(\mu_2 t_2 + \frac{1}{2}\sigma_2^2 t_2^2\right) + \left(\mu_3 t_3 + \frac{1}{2}\sigma_3^2 t_3^2\right)\right) \\ = M_{X_1}\left(t\right) M_{X_2}\left(t\right) M_{X_3}\left(t\right) \end{aligned} \tag{27}$$