Applied Sabbatical: A Discussion of Crop Insurance: Lecture IX

Charles B. Moss

September 2, 2026

Reading Question

- Derive the general form of the normal from the standard normal distribution

$$f\left(x\right)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right). \tag{1}$$

State Dependent Production Functions

- As a starting point, we must consider how randomness is "built into" most production function estimations. As I will discuss in Production Economics, this typically involves an error of convenience. For example, modeling production using a Cobb-Douglas production

$$f(x_1, x_2) = A x_1^{\alpha} x_2^{\beta} \tag{2}$$

- The typical approach is to estimate this by taking the logarithms of both sides

$$\ln(y) = \ln(A) + \alpha \ln(x_1) + \beta \ln(x_2) + \epsilon. \tag{3}$$

A basic question is what do we assume about ϵ? We can make several assumptions that make estimation easier – we can assume that $\epsilon \sim \mathrm{N}\left[0, \sigma^2\right]$.

- Another approach is model production as state-dependent following the general derivation of Chambers. and Quiggin (2000).
- As a simplified version of this concept, assume that production is a function of two "inputs" – $f(x,\omega)$.

$$
\begin{aligned}
f(x,\omega) = f(x_0,\omega) + \left.\frac{\partial f(x,\omega)}{\partial x}\right|_{x=x_0}(x-x_0) + \\
\left.\frac{\partial^2 f(x,\omega)}{\partial x^2}\right|_{x=x_0}(x-x_0)^2 + \epsilon(x,\omega) \\
= a(\omega) + b(\omega)x + c(\omega)x^2 \\
a(\omega) = f(x_0,\omega) - \left.\frac{\partial f(x,\omega)}{\partial x}\right|_{x=x_0} x_0 + \frac{1}{2}\left.\frac{\partial^2 f(x,\omega)}{\partial x^2}\right|_{x=x_0} x_0^2 \quad \cdot \qquad (4) \\
b(\omega) = \left.\frac{\partial f(x,\omega)}{\partial x}\right|_{x=x_0} - \left.\frac{\partial^2 f(x,\omega)}{\partial x^2}\right|_{x=x_0} x_0 \\
c(\omega) = \frac{1}{2}\left.\frac{\partial^2 f(x,\omega)}{\partial x^2}\right|
\end{aligned}
$$

- Next, if we restrict the random variable to a Bernoulli distribution $a(\omega) \rightarrow \{\, a(\omega_1), a(\omega_2)\}$, $b(\omega) \rightarrow \{\, b(\omega_1), b(\omega_2)\}$, $c(\omega) \rightarrow \{\, c(\omega_1), c(\omega_2)\}$, $\mathrm{P}[\omega_1] = \theta$, and $\mathrm{P}[\omega_2] = 1 - \theta$, we derive a stochastic (state-dependent) production function for two specific outcomes

$$f(x,\omega) = \begin{cases} a(\omega_1) + b(\omega_1)\, x + c(\omega_1)\, x^2 \\ a(\omega_2) + b(\omega_2)\, x + c(\omega_2)\, x^2 \end{cases} \tag{5}$$

- For those in Research Methodology, remember the production function from Bareille and Chakir

$$y_{i,j,t} = \alpha_j\left(w_{i,t}\right) - \frac{1}{2}\sum_{k=1}^{2}\sum_{l=1}^{2}\gamma_{j,k,l}\left(w_{i,t}\right)\left[\beta_{j,k}\left(w_{i,t}\right) - x_{i,j,k,t}\right]\left[\beta_{j,l}\left(w_{i,t}\right) - x_{i,j,l,t}\right] \tag{6}$$

- In a two-input model

$$g\left(x_1, x_2, \omega\right) = a_0\left(\omega\right) + \left[\begin{array}{c} a_1\left(\omega\right) \\ a_2\left(\omega\right) \end{array}\right]'\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] + \frac{1}{2}\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]'\left[\begin{array}{cc} A_{11}\left(\omega\right) & A_{12}\left(\omega\right) \\ A_{12}\left(\omega\right) & A_{22}\left(\omega\right) \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \tag{7}$$

- Suppose that the choice of x_2 (fertilizer) (i.e., $x_2 = \tilde{x}_2$) is fixed, the production function would then be written as

$$\begin{aligned} g\left(x_1, \tilde{x}_2, \omega\right) = &\left[a_0\left(\omega\right) + a_2\left(\omega\right)\tilde{x}_2 + \frac{1}{2}A_{22}\left(\omega\right)\tilde{x}_2^2\right] + \\ &a_1\left(\omega\right)x_1 + \frac{1}{2}A_{11}\left(\omega\right)x_1^2 + A_{12}\left(\omega\right)x_1\tilde{x}_2 \\ = &\left(a_1\left(\omega\right) + A_{12}\left(\omega\right)\tilde{x}_2\right)x_1 + A_{11}\left(\omega\right)x_1^2 + \epsilon\left(\omega\right) \\ \epsilon\left(\omega\right) = &\, a_0\left(\omega\right) + a_2\left(\omega\right)\tilde{x}_2 + \frac{1}{2}A_{22}\left(\omega\right)\tilde{x}_2^2 \end{aligned} \tag{8}$$

- I will solve for an approximation of the production function using the profit maximization conditions under certainty. Specifically, I will assume that

$$
\begin{gathered}
a(\omega) = 0 \\
b(\omega), c(\omega) \Leftarrow \left\{ \begin{array}{c} x(\omega) = \left[\frac{p_x}{p_y} - b(\omega)\right] \frac{1}{2c(\omega)} \\ y(\omega) = b(\omega)\, x(\omega) + c(\omega)\, x(\omega)^2 \end{array} \right. . \\
\ni : \omega \{\, \omega_1, \omega_2 \} \;\; x(\omega) = \{\, x_1^*, x_2^* \} \;,\; y(\omega) = \{\, y_1^*, y_2^* \}
\end{gathered}
\tag{9}
$$

If we set $p_y = 3.50$, $p_x = 250$, and in state 1 $x_1^* = 55$ and $y_1^* = 125$ Equation 9 implies that $b(\omega) = 4.6097$ and $c(\omega) = -0.0407$. Similarly, if $x_2^* = 65$ in state 2 and $y_2 = 145$, Equation 5 implies that $b(\omega_1) = 4.4258$ and $c(\omega_2) = -0.0338$.

- The expected value of the production function then becomes

$$\begin{aligned} \mathrm{E}\left[f\left(x,\omega\right)\right] = 0.5\left[4.5097x - 0.0407x^2\right] + \\ (1-0.5)\left[4.4258x - 0.0338x^2\right] \\ = 4.4678x - 0.0372x^2 \end{aligned} \quad (10)$$

Consequences of Crop Insurance

- As a starting point, assume that the farmer simply maximizes expected profit

$$\max_x \left[4.4678x - 0.0372x^2\right] 3.50 - 0.125x \tag{11}$$

where we assume that the price of output is \$ 3.50 per bushel and the cost of fertilizer is \$ 0.125 per pound. The result is $\pi^* = 461.41$ and $x^* = 59.4904$.

- Next, we define an insurance contract

$$\phi\left(x, p_y\right) = \mathrm{Max}\left(y^* - f\left(x\right), 0\right) p_y = \begin{cases} \left(y^* - f\left(x\right)\right) p_y, \text{ if } f\left(x\right) < y^* \\ 0 \text{ if } f\left(x\right) \geq y^* \end{cases} \tag{12}$$

where $\phi\left(\right)$ defines the indemnity payment in dollars.

- If we set $y^* = 125$ and assume that the farmer still uses $x^* = 59.4904$, the yield in state 1 is 124.2422 bushels and in state 2 is 143.6708 bushels. Therefore, the insurance policy pays \$ 2.364 per acre in state 1.
- From a slightly different perspective, the farmer receives

$$R\left(y^*, x, p_y\right) = \mathrm{Max} f\left(y^*, f\left(x\right)\right) p_y = \begin{cases} y^* p_y \text{ if } f\left(x\right) \leq y^* \\ f\left(x\right) p_y \text{ if } f\left(x\right) \geq y^* \end{cases} \tag{13}$$

where $R\left(\right)$ is the revenue function.

- The revenue function can be rewritten slightly as

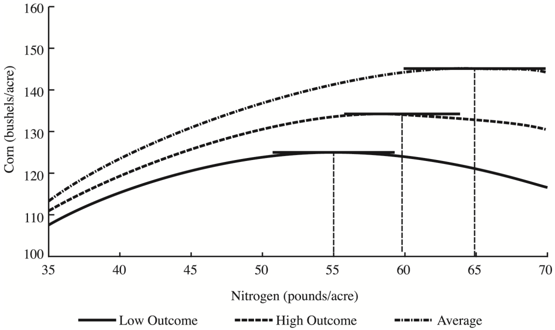
$$f(x)\,p_y + \phi(x, p_y) = \left\{ \begin{array}{c} f(x)\,p_y + (y^* - f(x))\,p_y = y^* p_y \text{ if } f(x) < y^* \\ f(x)\,p_y + 0 \text{ if } f(x) \geq y^* \end{array} \right. . \quad (14)$$

Based on this formulation, the firm's maximization problem becomes

$$\begin{gathered} \pi = \max x 0.5 \left[\mathrm{Max}\left(4.5097x - 0.0407x^2, 125\right) 3.50 - 0.125x \right] + \\ 0.5 \left[\mathrm{Max}\left(4.4258x - 0.0338x^2, 125\right) 3.50 - 0.125x \right] - \phi \end{gathered} \quad (15)$$

where ϕ is the cost of the insurance policy.

- Note that even if we make the insurance policy actually sound (i.e., $0.5 \times 2.364 = 1.326$), the existence of the insurance payoff distorts the optimum level of fertilizer which becomes $x^{*} = 64.414$.



The Actual Crop Insurance Contract

- The implementation of the current crop insurance program is based on an expected or average level of payment.
  - To start our discussion, we will assume that the proven average yield for soybeans in Alachua county Florida is 45 bushels per acre.
  - My previous results indicate that soybean yield tend to be normally distributed Ramirez et al. (1994).
  - So let's assume that $y_S \sim \mathrm{N}(45, 100)$. Following the notions in Equation 14, we want to set the insured level of yield.

- Under the current program, this is typically expressed as a percentage of the average yield. Also in the program, the USDA will pay 80 % of the cost of insurance for 50 % coverage. Let us begin by asking a simple question – what is the likelihood that this instrument will pay?

$$\mathrm{P}\left[y_S \leq 0.50 \times 45\right] = \int_{-\infty}^{0.50\times 45} f\left(z|\mu = 45, \sigma^2 = 100\right) dz = 0.0122 \tag{16}$$

  so the probability that this insurance policy pays off is relatively small.
- To make this problem more interesting, let us consider a payoff trigger of 80 % which payoff with probability 0.1841.
- Given this payoff limit – what does the insurance product pay?

$$\phi = 3.50 \int_{-\infty}^{0.80\times 45} \left(0.80 \times 45 - z\right) f\left(z|\mu = 45, \sigma^2 = 100\right) dz = 3.51464 \tag{17}$$

- Most of my research in this area has dealt with the non-normality of yield distributions. Specifically, I have been interested in deviations in the third and fourth moments of the yield distribution.
- Moss and Shonkwiler (1993) use a transformation to normality referred to as the Inverse Hyperbolic Sine Transformation

$$\begin{gathered}\epsilon=\frac{\ln\left(\theta z+\sqrt{1+\theta^2 z^2}\right)}{\theta} \\ \nu\sim\mathrm{N}\left[\delta,\sigma^2\right].\end{gathered} \tag{18}$$

- This transformation introduces the potential for skewness and leptokurtosis (fat tails).

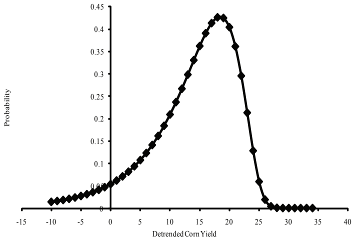


Figure: Inverse Hyperbolic Sine Distribution

References

Chambers., R. G. and Quiggin, J. (2000). *Uncertainty, Production, Choice, and Agency*. Cambridge University Press.

Moss, C. B. and Shonkwiler, J. (1993). Estimating yield distributions using a stochastic trend model and nonnormal errors. *American Journal of Agricultural Economics*, 75(5):1056–1062.

Ramirez, O. A., Moss, C. B., and Boggess, W. G. (1994). Estimation and use of the inverse hyperbolic sine transformation to model nonnormal correlated random variables. *Journal of Applied Statistics*, 21:289–304.