Derivation of the Normal Distribution: Lecture VIII

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Reading Question

- Assume that $f(x) = 1$ for $0 \leq x \leq 1$. What is the probability density function for the transformed variable $y = \sqrt{x}$?

Polar Integral Form

- The trick to this integration is changing the variables into a polar form.
  - *Polar Integration*: The notion of polar integration is basically one of a change in variables. Specifically, some integrals may be ill-posed in the traditional Cartesian plane, but easily solved in a polar space.
  - By polar space, any point $y = f(x)$ can be written in a trigonometric form

$$
\begin{gathered}
r = \sqrt{x^2 + y^2} \\
\theta = \tan^{-1}(x/y) \\
y = r\cos(\theta) \\
x = r\sin(\theta)
\end{gathered} \tag{1}
$$

Example Polar Map

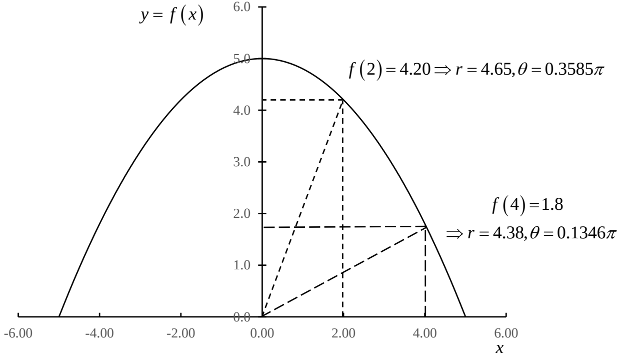
- Start with a quadratic function

$$y = f(x) = 5 - \frac{x^2}{5}\, x \in [-5, 5] \tag{2}$$

- Consider the point $f(4.0) = 1.8$.
- The length of the ray from the origin is

$$r(4, 1.8) = \sqrt{4^2 + 1.8^2} = 4.38 \tag{3}$$

Example Plot



Inscribed Angle

- We can compute the value of the inscribed angle

$$\tan\left(\theta\left(4,1.8\right)\right) = \frac{1.8}{4} = 0.45. \tag{4}$$

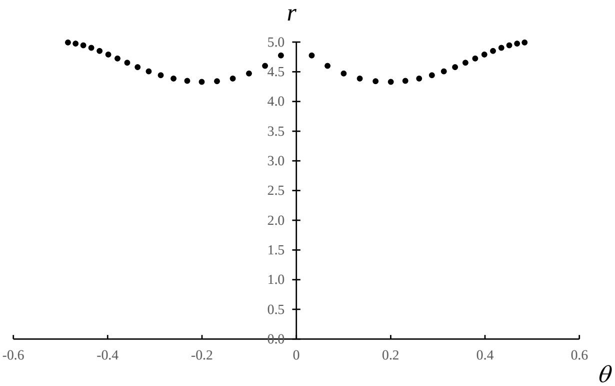
- The question is then: What angle has a $\tan$ of 0.45

$$\theta\left(4,1.8\right) = \tan^{-1}\left(0.45\right) = 0.1346\pi. \tag{5}$$

- Basically, we are transforming the old functional relationship $f\left(x,y\right) = 0$ into a new relationship $g\left(\theta, r\left(\theta\right)\right)$. Hence

$$r\left(\theta\right) = 4.38. \tag{6}$$

Polar Transformation of Simple Quadratic



Derivation of the Normal

- The order of proof of the normal distribution function is to start with the standard normal:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{7}$$

- First, we need to demonstrate that the distribution function does integrate to one over the entire sample space, which is $-\infty$ to ∞. This is typically accomplished by proving the constant.

- Let us start by assuming that

$$I = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \tag{8}$$

Squaring this expression yields

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \\ &= \int_{-\infty}^{\infty} e^{-\frac{y^2+x^2}{2}} dy dx \end{aligned} \tag{9}$$

- By substituting $y = r\cos(\theta)$ and $x = r\sin(\theta)$ we derive

$$\begin{aligned} y^2 + x^2 &= r^2\cos^2(\theta) + r^2\sin^2(\theta) \\ &= r^2\left[\cos^2(\theta) + \sin^2(\theta)\right] \\ &= r^2 \end{aligned} \tag{10}$$

Defining the derivatives of the change in variables

$$dydx = rdrd\theta \tag{11}$$

Folding these two results together we get

$$I^2 = \int_0^{2\pi}\int_0^{\infty} r\exp\left(-\frac{r^2}{2}\right)drd\theta = \int_0^{2\pi} d\theta = 2\pi \tag{12}$$

- Backing up a little bit - consider the derivative

$$\frac{\partial \exp\left(-\frac{r^2}{2}\right)}{\partial r} = -r\exp\left(-\frac{r^2}{2}\right). \tag{13}$$

Therefore

$$\int r\exp\left(-\frac{r^2}{2}\right)dr = -\exp\left(-\frac{r^2}{2}\right) \tag{14}$$

- Over the interval

$$\left(-\exp\left(-\frac{r^2}{2}\right)\right|_0^\infty \to -\exp\left(-\frac{\infty^2}{2}\right)+\exp\left(-\frac{0^2}{2}\right)=1 \tag{15}$$

- Taking the square root of each side yields

$$I = \sqrt{2\pi} \tag{16}$$

Thus, we know that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1 \tag{17}$$

- The expression above is the expression for the standard normal. A more general form of the normal distribution function can be derived by defining a transformation function. Defining

$$\begin{aligned} y &= a + bx \\ x &= \frac{y-a}{b} \end{aligned} \tag{18}$$

By the change in variable technique, we have

$$f\left(x\right) = \frac{1}{\sqrt{2\pi}} e^{-\dfrac{\left(y-a\right)^2}{2b^2}} \left|\frac{1}{b}\right| \tag{19}$$

Expected Values

- **Definition 2.2.1.** The expected value or mean of a random variable $g(X)$ denoted $\mathrm{E}[\mathrm{g}(\mathrm{x})]$ is

$$\mathrm{E}[\mathrm{g}(\mathrm{X})] = \begin{cases} \int_{\infty}^{\infty} g(x) f(x) dx \text{ if } x \text{ is continuous} \\ \sum_{x \in X} g(x) f_x \text{ if } x \text{ is discrete} \end{cases} \tag{20}$$

- The most general form used in this definition allows for taking the expectation of the function $g(X)$. Strictly speaking, the mean of the distribution is found where $g(X) = X$, or

$$\mathrm{E}[x] = \int_{-\infty}^{\infty} x f(x) dx \tag{21}$$

- For example, we take the exponential distribution

$$
\begin{aligned}
\mathrm{E}\left[x\right] = \int_0^\infty \frac{1}{\lambda} x e^{-\frac{x}{\lambda}} dx \\
= -\left(x e^{-\frac{x}{\lambda}} \right|_0^\infty + \int_0^\infty e^{\frac{x}{\lambda}} dx \\
= -\left(\lambda e^{-\frac{x}{\lambda}} \right|_0^\infty = \lambda
\end{aligned}
\tag{22}
$$

- **Theorem 2.2.1.** Let X be a random variable and let a, b, and c be constants. Then for any functions $g_1(X)$ and $g_2(X)$ whose expectations exist:
    - $\mathrm{E}[ag_1(X) + bg_2(X)] = a\mathrm{E}[g_1(X)] + b\mathrm{E}[g_2(X)] + c$.
    - If $g_1(X) \geq 0$ for all X, then $\mathrm{E}[g_1(X)] \geq 0$.
    - If $g_1(X) \geq g_2(X)$ for all X, then $\mathrm{E}[g_1(X)] \geq \mathrm{E}[g_2(X)]$.
    - If $a \leq g_1(X) \leq b$ for all X, then $a \leq \mathrm{E}[g_1(X)] \leq b$.

Reading for Next Time

- Moss, Charles B. and J. S. Shonkwiler. 1993. Estimating Yield Distributions Using a Stochastic Trend Model and Nonnormal Errors. *American Journal of Agricultural Economics* 75(4), 1056–62. [DOI: 10.2307/1243993, JSTOR: 1243993](Google Scholar – 210, Web of Science – 69, Scopus – 84)
- Ramirez, Octavio A., Charles B. Moss, and William G. Boggess. 1994. Estimation and Use of the Inverse Hyperbolic Sine Transformation to Model Non-Normal Correlated Random Variables. *Journal of Applied Statistics* 21: 289-304. [DOI: 10.1080/757583872] (Web of Science – 10, Scopus – 15)